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ADAPTATION IN FEEDBACK CONTROL SYSTEMS

BY

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ABSTRACT

The scope of the word "adaptive" in the literature of feedback control systems is discussed. Approaching the subject from biological adaptation, the author shows that any control system which makes effective use of feedback is adaptive, and therefore recommends that "adaptive" should not be used to denote a separate class of feedback control systems.

INTRODUCTION

The word "adaptive" has recently become prominent in the literature of feedback control systems. It is generally supposed that, through the study of adaptive control systems, ideas from adaptive processes in biological systems can be applied to the solution of problems in engineering control systems. Unfortunately, however, papers on the subject of adaptive control systems have not given a clear and unequivocal definition of what adaptation in a control system really is. The purpose of this paper is to discuss the meaning of the term, with a view to clarifying its proper usage in the literature.

ADAPTATION IN BIOLOGY

Since it is generally assumed that the term "adaptive" is used in a sense similar to that in which it is used in biology (1, 2),[2] it is logical to begin by studying the literature of biology in order to discover the essential characteristics of adaptive behavior. One definition has been given by Ashby (3), who states that "a form of behaviour is adaptive if it maintains the essential variables within physiological limits." It seems obvious from this definition that any control system which makes

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[2] The boldface numbers in parentheses refer to the references appended to this paper.

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effective use of feedback can be described as being adaptive. The definition adopted, however, should not depend on one opinion alone, and so a typical adaptive process in biology will be considered in order to determine what are the essential characteristics of adaptation in such a situation.

Suppose, for example, the process of temperature control of warm-blooded animals is considered. One particular aspect of this control process serves as an illustration of what is meant by adaptation, and this will now be discussed. A fur-bearing animal can control the rate of heat dissipation from its body by erecting or relaxing the hairs of its fur; this process is describable in terms of a simple feedback control system. Furthermore, it is known that animals which live in arctic climates grow heavier and thicker coats of fur than those of similar species which live in more temperate climates; this is certainly an adaptation, where "adaptation" is defined according to Webster (4) as "modification of an animal or plant (or of its parts or organs) fitting it more perfectly for existence under the conditions of its environment."

It is the author's contention that these two processes, the simple adjustment of existing fur and the adaptive growth of a more suitable coat of fur, are essentially the same from the control engineer's point of view. The control engineer is not concerned with whether or not any changes occur in the animal's weight or appearance, but only with whether or not these changes affect its body temperature; he will therefore view each of these processes as a change in the thermal insulation between the animal's body and its environment, the fact that one process is more permanent than the other being merely a difference in time scale.

The study of biological adaptation therefore indicates that any feedback control system may be described as being adaptive. It is, however, desirable to examine the field of control systems in a general way in order to determine whether the term "adaptive" is relevant or not, without any preconceived ideas as to what is meant by adaptation in any other discipline.

ADAPTATION IN CONTROL SYSTEMS

It is obvious that adaptation must involve change, but it is equally obvious that control of any kind must involve change. The basic question therefore is: what kind of change is adaptive, and what kind is not? Authors of previous papers on the subject have answered this question by saying that a parameter change is adaptive, whereas a change in signal is not. There appear to be two basic objections to this somewhat arbitrary separation of types of change. In the first place, if a quantity in a control system is automatically and continuously adjustable, it is highly doubtful that it can in any real sense retain the name of parameter. In the second place, the variation of parameter

can only be effected by the application of a suitable signal at the appropriate point in the system.

An "adaptable" transducer is generally represented in a diagram as a block with two inputs and one output (2). One of the inputs is the conventional input signal, whereas the other input is a signal, derived from some suitable source, which varies the transfer characteristics of the transducer. If an external observer studied the behavior of that part of the system by observing the conventional input variable and the output he would find the behavior indeterminate, and if the indeterminacy were beneficial he might say that the behavior was adaptive. On the other hand, if the observer studied its operation as a two-input one-output device he would find it quite determinate. Furthermore, the observer would notice that if either input was kept fixed at a particular value there would be a definite transfer relationship between the other input and the output, and if he had no further knowledge he could not say which was the parameter-varying input. The system can be thought of as a one-input device with an adaptive feature or as a two-input device with no adaptation. If the two-input case is chosen, the source from which the "parameter-varying" signal comes is merely a compensator. That is to say, whether the control is called adaptive or not depends on the system chosen, where *system* is defined according to Ashby (3) as "any arbitrarily chosen set of variables."

In short, the only noteworthy feature about the control systems which have been called adaptive seems to be that they automatically vary some arbitrary quantities which happen to remain fixed in the conventional feedback control system. It seems very doubtful that this is sufficient ground for glorifying the new class of control system with an adjective which implies a higher level of control. In any case, the control problem is rarely amenable to these methods. The usual situation is that one has a transducer or process which has an unalterable transfer function, the designer's problem being to synthesize a compensating device which will allow the desired over-all performance to be achieved. Therefore any apparent variations of parameters will be in the controller or compensator; a controller which behaves in such a way is described by Johnson (5) as an adaptive controller. However, the controller is merely a device which operates on certain input variables in such a way as to produce a desired set of output variables, although its operation may be described by saying that its characteristics are alterable in accordance with certain conditions. The alterable characteristics must be adjusted, directly or indirectly, by the inputs to the controller, so that the output is a function of the present and past values of the input variables. The controller may operate as a sequential transducer, but it is doubtful whether this feature makes the controller adaptive.

In June, 1957, the author contributed to the discussion of a paper on

"Adaptive Servomechanisms" (1), criticizing the definition of "adaptive" for reasons similar to those outlined in this paper; that is, that any feedback control system may be said to be adaptive. In April, 1958, the author was present at a panel discussion on the subject of "Control Systems which Automatically Optimize Performance," which degenerated into a discussion of the meaning of "adaptive," and which showed that the increase in the number of papers on the subject in the intervening months had tended to increase the confusion in the terminology. In the report (6) on the conference at which this panel discussion took place, no mention is made of the subjects discussed by the panel, though the report does quote a delegate from the U.S.S.R. as saying in an interview that "In the Soviet Union, as here, there is the argument as to what is 'self-adaptive, self-adjusting.'" It seems surprising that so little notice appears to be taken of the importance of formulating a satisfactory definition of adaptive behavior in control systems.

Aseltine *et al.* (2) have discussed the usage of "adaptive" by listing the various types of adaptation which have been studied. Their first category is Passive Adaptation, and the opening sentences of their discussion of this category are as follows:

"The classical example of passive adaptation is feedback itself. The reduction of effects of element changes through the use of feedback is well known. There are, however, special configurations which have inherent adaptive behavior beyond that found in conventional feedback systems.

"The degree of adaptive behavior in a linear feedback system can be increased by prefiltering the input signal appropriately."

In other words, the authors are saying, the so-called adaptive systems differ from the conventional feedback control system in degree but not in kind; this means that if the class of adaptive control systems is treated as a sub-class of all feedback control systems the dividing line can only be chosen arbitrarily.

Drenick and Shahbender (1), in their rebuttal to the author's discussion of their definition of "adaptive," have given the following list of terms which have been used to describe systems similar to those described by them: "input-controlled," "sympathetic," "learning," "self-adjusting," "self-designing," and "adaptive." It is not the purpose of this paper to discuss all the possible terms which might be applied to such systems, except to suggest that the use of six different terms to mean the same thing is not conducive to the clarity of thought and precision of expression which are essential to the dissemination and furthering of scientific knowledge.

CONCLUSION

Because of the foregoing arguments, it is the conclusion of the author that "adaptive" should not be used as a technical term to specify

a special class of feedback control systems. However, the word may be quite appropriate as a descriptive term applicable to *all* feedback control systems, possibly a better word than "feedback" itself. The only thing the word "feedback" tells one about a system is that it has a certain kind of topographical representation; on the other hand, "adaptive" tells one something about its function, which is what primarily interests the control engineer. In other words, the significant thing about the feedback system is not that it happens to possess a feedback loop, but that it possesses the ability to *adapt* to disturbances and to variations in the parameters of the forward loop. All control systems which make effective use of feedback are adaptive; therefore the word "adaptive" should not be used to describe an arbitrarily-selected class of such systems.

It is hoped that this discussion of the subject will help to clarify the proper usage of the word "adaptive," and that some attempt will be made to establish a consistent and logical terminology which will minimize ambiguity and confusion.

Acknowledgments

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REFERENCES

(1) R. F. DRENICK AND R. A. SHAHBENDER, "Adaptive Servomechanisms," *Trans. AIEE*, Vol. 76, Part II, "Applications and Industry," pp. 286–292 (1957).
(2) J. A. ASELTINE, A. R. MANCINI AND C. W. SARTURE, "A Survey of Adaptive Control Systems," *IRE Trans.*, PGAC-6, pp. 102–108 (1958).
(3) W. Ross ASHBY, "Design for a Brain," New York, John Wiley & Sons, Inc., 1954.
(4) Webster's New Collegiate Dictionary (1956).
(5) CHARLES W. JOHNSON, "Adaptive Servomechanisms," MH Aero Document R-ED 6101, Minneapolis-Honeywell Regulator Company, Minneapolis, Minn.
(6) "Fourth IRD Conference Advances 'Automatic Optimization,'" *Mech. Engineering*, Vol. 80, No. 6, pp. 130–131 (1958).

MODES AND NATURAL FREQUENCIES OF SUSPENSION BRIDGE OSCILLATIONS

BY

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SYNOPSIS

The natural frequencies of potential modes of suspension bridge oscillations constitute basic data required in studies for aerodynamic stability.

The formulas presented herein were derived by the author in 1941–1943 and have since been tested, for simplicity and practical usefulness, by fifteen or more years of application in the author's office practice. They have also been independently checked, for validity and accuracy, by others using more difficult formulas involving transcendental (complex or hyperbolic) functions. The author's formulas appear to be the simplest and most practical thus far developed for the calculation of the natural frequencies of all possible modes of suspension bridge oscillations.

For convenience of reference and application, these formulas are here assembled, derived and presented in definitive form, together with numerical examples to illustrate their practical application. The numerical examples are based on computations for the Mackinac Bridge.[2]

DEFINITIONS AND BASIC RELATIONS

If N is the natural frequency (cycles per second) of any harmonic oscillation, and ω is the circular frequency (radians per second), then

$$\omega = 2\pi N = \sqrt{\frac{K}{m}} \tag{1}$$

where $m = w/g$ is the oscillating mass (in the case of a span, m is the mass per unit span-length) and K is the "spring constant," or ratio of the elastic force of restitution (Δw), per unit span-length, to the displacement η at any point. Hence, by definition,

$$K = m\omega^2 = \frac{\Delta w}{\eta}. \tag{2}$$

K is also termed the "*Coefficient of Rigidity*" of the oscillating system, a significant criterion for aerodynamic stability.

For an oscillation of any span (l) in the form of a simple sine-curve (see Figs. 1 and 2), with n segments (half-waves) in the span, the amplitude η at any point x is given by:

$$\eta = a \sin \frac{\pi n}{l} x \tag{3}$$

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[2] EDITOR'S NOTE: See this JOURNAL, Vol. 262, p. 453 (1956).

where a is the amplitude at mid-segment. The displacement η_t at any instant t is

$$\eta_t = \eta \sin \omega t \tag{4}$$

for $\eta_t = 0$, at $t = 0$.

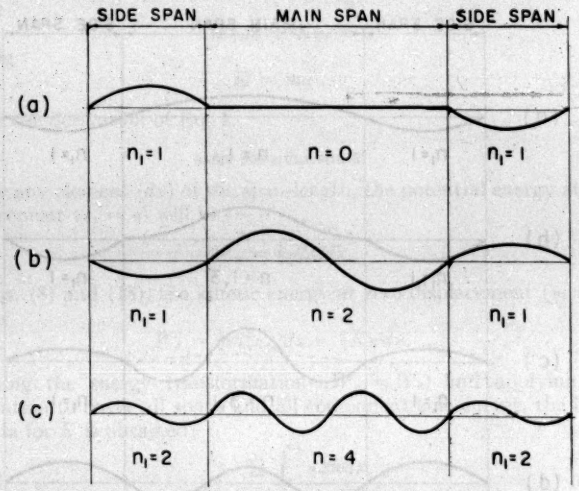

FIG. 1. Antisymmetrical modes.

In the derivations that follow, useful integrals are:

$$\int_0^l \eta dx = 0 \quad \text{(for } n \text{ even)} \tag{5}$$

$$\int_0^l \eta dx = \frac{2al}{\pi n} \quad \text{(for } n \text{ odd)} \tag{6}$$

$$\int_0^l \eta^2 dx = \tfrac{1}{2}a^2 l. \tag{7}$$

Useful differentials are:

$$\dot{\eta}_t = \frac{\partial \eta_t}{\partial t} = \omega\eta \quad (\text{at } \eta_t = 0) \tag{8}$$

$$\ddot{\eta}_t = \frac{\partial^2 \eta_t}{\partial t^2} = -\omega^2 \eta_t \tag{9}$$

$$\frac{d^2\eta}{dx^2} = -\frac{\pi^2 n^2}{l^2}\eta \tag{10}$$

$$\frac{d^4\eta}{dx^4} = \frac{\pi^4 n^4}{l^4}\eta. \tag{11}$$

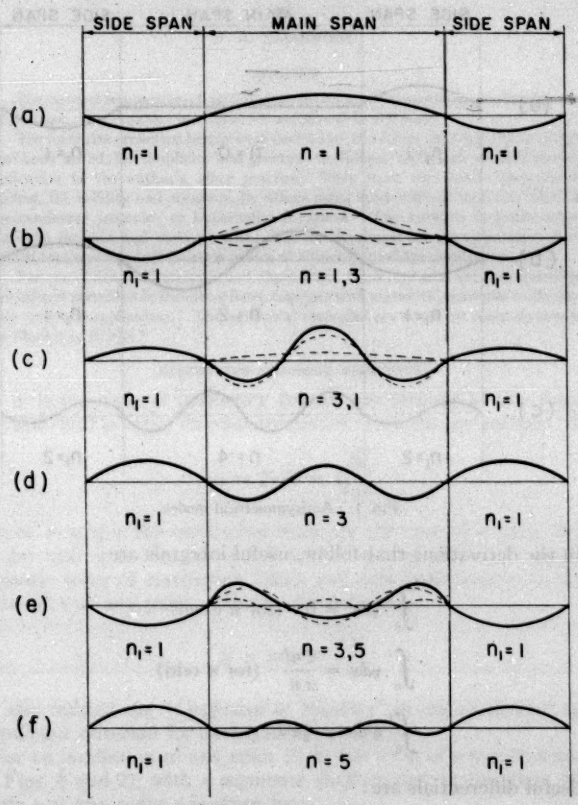

FIG. 2. Symmetrical modes.

The cable curve (with sag f) is assumed parabolic:

$$y = 4x(l - x)\frac{f}{l^2} \tag{12}$$

yielding

$$\frac{d^2y}{dx^2} = -\frac{8f}{l^2}. \tag{13}$$

By Eq. 9, the force equation at any instant on any unit length of span is

$$-\omega^2\eta_t m + K\eta_t = 0 \tag{14}$$

yielding

$$K = m\omega^2. \tag{15}$$

This is the derivation of Eq. 1.

BASIC FORMULA FOR K

For any element (dx) of the span-length, the potential energy at full displacement ($\eta_t = \eta$) will be

$$W_p = \tfrac{1}{2}\eta\Delta w dx. \tag{16}$$

By Eqs. (8) and (15), the kinetic energy at zero displacement ($\eta_t = 0$) will be

$$W_k = \tfrac{1}{2}m(\dot{\eta}_t)^2 dx = \tfrac{1}{2}K\eta^2 dx. \tag{17}$$

Equating the energy transformation ($W_p = W_k$) and applying the summation to cover all spans and all component sine-curves, the basic formula for K is obtained:

$$K = \frac{\Sigma\int_0^l \eta\Delta w dx}{\Sigma\int_0^l \eta^2 dx}. \tag{18}$$

If a span has a coupled synchronous *longitudinal* oscillation of amplitude e, then W_k, given by integration of Eq. 17, is augmented by

$$\Delta W_k = \tfrac{1}{2}m(\dot{e})^2 l = \tfrac{1}{2}m\omega^2 e^2 l = \tfrac{1}{2}Ke^2 l \tag{19}$$

and Eq. 18 becomes

$$K = \frac{\Sigma\int_0^l \eta\Delta w dx}{\Sigma\int_0^l \eta^2 dx + \Sigma e^2 l}. \tag{20}$$

GENERAL FORMULA FOR K

The fundamental equilibrium equation for a suspension bridge is

$$\Delta w = -\frac{d^2M}{dx^2} = -\frac{d^2}{dx^2}\left[H\eta + y\Delta H - EI\frac{d^2\eta}{dx^2}\right] \tag{21}$$

where H is the horizontal cable tension, ΔH is the increment produced by displacement, I is the moment of inertia of the section of the stiffening truss, and E is the modulus of elasticity for the truss.

By Eqs. 10, 11 and 13, Eq. 21 yields:

$$\Delta w = \frac{\pi^2 n^2}{l^2} H\eta + \frac{\pi^4 n^4}{l^4} EI\eta + \frac{8f}{l^2} \Delta H. \tag{22}$$

For any case in which $\Delta H = 0$, Eq. 2 or 18 then yields:

$$K = \frac{\pi^2 n^2}{l^2} H + \frac{\pi^4 n^4}{l^4} EI. \tag{23}$$

Equation 23 is the simplest and most useful formula for K. It is exact for all *even* values of n ($n = 2, 4, 6, \cdots$), the so-called "*antisymmetric modes*" (Fig. 1), since, for these modes, $\Delta H = 0$.

The value of K for $n = 2$ is the "*Coefficient of Rigidity*" of the suspension bridge, a useful criterion for aerodynamic stability.

In Eq. 23, the H-term and the EI-term represent the respective contributions of the cable and the stiffening truss to the rigidity of the suspension bridge. The ratio of the truss term to the total is designated by R and is significant for aerodynamic stability.

From Eqs. 20, 22, 6 and 7,

$$K = \frac{\pi^2 \Sigma \left(\frac{n^2 a^2}{l}\right) H + \pi^4 \Sigma \left(\frac{n^4 a^2}{l^3}\right) EI + \frac{32}{\pi} \Sigma \left(\frac{af}{nl}\right) \Delta H}{\frac{1}{w_0} \Sigma (wa^2 l) + \frac{2}{w_0} \Sigma (we^2 l)}. \tag{24}$$

This is the most general form of the equation for K. It covers all possible combinations of spans, modes, and superimposed harmonics.

In Eq. 24 and the following equations, and in all similar formulas involving summations of functions of the amplitudes, it is not necessary to know the absolute magnitudes of the several concurrent amplitudes (a and e), but merely the relative magnitudes.

Equation 24 is completely general for any number of spans of any respective lengths (l), simple or continuous, with any respective cable sags (f), including straight or unloaded backstays, for any combinations of values of n and a in simple or superimposed harmonic oscillations in the different spans, with any concomitant longitudinal oscillations (e) in any spans, with different values of w, H, EI, in different spans, and with any value of ΔH producing harmonic cable stretch.

In the denominator of Eq. 24, w_0 is the value of w for the main span.

Example 1

Antisymmetric Modes—K by Eq. 23
(Fig. 1)

For the Mackinac Bridge:

$l = 3800$ ft. $\qquad l_1 = 1800$ ft.

$w = 9400$ lb./ft.

$H = 48{,}478{,}000$ lb. (for two cables)

$EI = 3{,}768 \times 10^9$ lb. ft.^{2} (for two trusses)

(K in lb./ft.^{2})

	By Eq. 23	By Eq. 1
For $n = 2$,	$K = 132.5 + 28.2 = 160.7$	$N = 7.09$/min.
$n = 4$,	$K = 530.1 + 450.6 = 980.7$	$N = 17.50$/min.
$n_1 = 1$,	$K = 147.7 + 35.0 = 182.7$	$N = 7.55$/min.
$n_1 = 2$,	$K = 590.7 + 559.4 = 1150.1$	$N = 18.95$/min.

EVALUATION OF ΔH

The virtual work represented by the vertical displacements η is

$$W_1 = \Sigma \frac{8f}{l^2} H \int_0^l \eta dx. \tag{25}$$

The virtual work represented by the cable stretch due to ΔH is

$$W_2 = \Sigma H \frac{\Delta H}{E_c A_c} \int_0^l \frac{ds^3}{dx^2}. \tag{26}$$

Hence, for $W_1 = W_2$, with Eq. 6,

$$\Delta H = \frac{16}{\pi} \frac{E_c A_c}{L_s} \Sigma' \left(\frac{af}{nl} \right) \tag{27}$$

where

$$L_s = \Sigma \int_0^l \frac{ds^3}{dx^2} \approx \Sigma \left(l + \frac{8f^2}{l} \right) \sec^3 \gamma. \tag{28}$$

In these equations, and those that follow, E_c is the elastic modulus and A_c the section of the cable, and γ is the inclination of the cable-chord. Also, Σ denotes summations covering all similar expressions in all spans, and Σ' denotes a summation covering only the *odd* values of n in the several spans.

COMPOSITE MODES

A simple or "*primary mode*" is one represented by a single sine-curve—hence, single values of n and of a, as in Eq. 23. The term

"*composite mode*" is used herein to denote oscillations comprising a plurality (superimposed or coupled, or both) of simple modes, as in Eq. 24 (see Fig. 2).

From Eq. 24, using the expressions of Eqs. 23 and 27, the general formula for K for any *single* component mode (any simple sine-curve) is

$$K = K_n + C\frac{1}{na}\Sigma'\left(\frac{af}{nl}\right) \tag{29}$$

where

$$C = \frac{512}{\pi^2}\frac{f}{l^2}\frac{E_c A_c}{L_s}, \tag{30}$$

a dimensional constant for the structure, and K_n is the value of K given by Eq. 23. For $\Sigma' = 0$, Eq. 29 reduces to Eq. 23.

For symmetric modes (necessarily composite), as illustrated in Fig. 2, Eq. 29 is valid for each component mode; and, with the consistent (normal) relative amplitudes substituted, it must yield the same value of K (and, hence, the same frequency N) for each component mode. To satisfy this condition, Eq. 29 yields two simple governing relations between K and the respective values of n, a, and K_n of the component modes:

$$\Sigma'\left[\frac{1}{n^2}\frac{C\frac{f}{l}}{(K - K_n)}\right] = 1 \tag{31}$$

and

$$na \propto \frac{1}{K - K_n}. \tag{32}$$

Equation 31 yields the desired value of K for the normal (balanced) composite mode, and Eq. 32 gives the relative amplitudes of the component modes.

In practical application, the largest number of component modes necessary to be considered and substituted in Eq. 31 is three, and that is the case of the lowest symmetric mode ($n = 1, 3$; and $n_1 = 1$, Fig. 2*b*). In the higher symmetric modes, the superimposed harmonics prove negligible, and only two primary K_n-values need to be substituted, one for each span (Example: $n = 3$ and $n_1 = 1$, Fig. 2*d*). Accordingly, Eq. 31 is in reality a quadratic, or cubic, equation for K. Each real root of this equation will represent a *normal* composite mode with a mathematically and physically consistent set of relative amplitudes.

In practical application, Eq. 31 lends itself to a simplified numerical solution by an expedited method of trial substitution. It is the sum of two or three fractions equated to unity, with the sole unknown, K, in each denominator. If the mode represented by the first fraction is dominant, substitute an approximate trial value of K in the other term

or terms, and solve the first fraction for K. The first or second trial will generally suffice. Similarly, the one or two other roots are obtained by treating the other fraction or fractions, respectively, as dominant.

The subscripts in a_1, n_1, f_1, etc., designate values for the side spans.

Example 2

Symmetric Composite Modes—K by Eq. 29

Mode: $n = 1, 3$; $n_1 = 1$

(Figs. 2*b*, 2*c*)

Components:	$n = 1$	$n = 3$	$n_1 = 1$
Amplitudes:	a	a_3	a_1
K_n (by Eq. 23):	34.9	440.8	182.7

(Compare Example 1.)

For the Mackinac Bridge:

$A_c = 744$ sq. in. $\qquad L_s = 9140$ ft.

$C = 2866$ lb/ft^2

$$C\frac{f}{l} = 264.0 \qquad C\frac{f_1}{l_1} = 125.0$$

By Eq. 31:

$$\frac{264.0}{K - 34.9} + \frac{264.0/9}{K - 440.8} + \frac{2(125.0)}{K - 182.7} = 1$$

Three solutions:

Dominant	K	Eq. 32
$n = 1$	$K = 99.5$	$a_3/a = -0.063$
	$(N = 5.57)$	$a_1/a = -0.776$
$n = 3$	$K = 405.6$	$a/a_3 = -0.285$
	$(N = 11.26)$	$a_1/a_3 = -0.474$
$n_1 = 1$	$K = 696.7$	$a/a_1 = 0.777$
	$(N = 14.75)$	$a_3/a_1 = 0.670$

Example 3

Mode: $n = 3, 5$; $n_1 = 1$ (Fig. 2*e*)

Components:	$n = 3$	$n = 5$	$n_1 = 1$
Amplitudes:	a_3	a_5	a_1
K_n (by Eq. 23):	440.8	1928.6	182.7

Eq. 31: $$\frac{264.0/9}{K - 440.8} + \frac{264.0/25}{K - 1928.6} + \frac{2(125.0)}{K - 182.7} = 1$$

Three solutions:

Dominant	K	Eq. 32
$n = 3$	363.2	$a_1/a_3 = -1.290$
	$(N = 10.65)$	$a_5/a_3 = 0.030$
$n = 5$	1941.2	$a_3/a_5 = 0.014$
	$(N = 24.62)$	$a_1/a_5 = 0.036$
$n_1 = 1$	537.6	$a_3/a_1 = 1.222$
	$(N = 12.96)$	$a_5/a_1 = -0.051$

For $n = 5$, dominant, the small value of a_3/a_5 indicates that the superposed harmonic $n = 3$ is negligible. Similarly, for $n = 3$, dominant, the superposed harmonic $n = 5$ is negligible.

Example 4

Mode: $n = 3, n_1 = 1$ (Fig. 2*d*)

Amplitudes:	a_3	a_1
K_n (by Eq. 23):	440.8	182.7

Eq. 31: $$\frac{264.0/9}{K - 440.8} + \frac{2(125.0)}{K - 182.7} = 1$$

Two solutions:

Dominant	K	Eq. 32
$n = 3$	363.8	$a_1/a_3 = -1.276$
	$(N = 10.66)$	
$n_1 = 1$	539.1	$a_3/a_1 = 1.209$
	$(N = 12.98)$	

Note that these results are practically identical with those obtained in Example 3. The inclusion of the third component ($n = 5$) in Example 3 was unnecessary; the effect is almost imperceptible.

Example 5

Mode: $n = 5, n_1 = 1$ (Fig. 2*f*)

Amplitudes:	a_5	a_1
K_n (by Eq. 23):	1928.6	182.7

Eq. 31: $$\frac{264/25}{K - 1928.6} + \frac{2(125.0)}{K - 182.7} = 1$$

Two solutions:

Dominant	K	Eq. 32
$n = 5$	1940.9	$a_1/a_5 = 0.035$
	$(N = 24.62)$	
$n_1 = 1$	430.9	$a_5/a_1 = -0.033$
	$(N = 11.60)$	

These results for $n = 5$ are practically identical with those obtained in Example 3.

For $n = 3, 5, \cdots$, with $n_1 = 1$, the inclusion of a third component in applying Eqs. 29–32 is unnecessary.

STRAIGHT OR UNLOADED BACKSTAYS

For the case of *straight* backstays, assuming zero sag ($f_1 = 0$), all of the foregoing equations, including Eqs. 29–32, are applicable without any modification. The sole effect is in the reduced numerical value of L_s given by Eq. 28, reflected in the increased numerical value of C, given by Eq. 30.

For application to *unloaded* backstays with the small cable-sag (f_1) considered, Eqs. 29, 31 and 32 require a simple modification. Instead of the usual relation of equal unit dead loads in main and side-spans ($w_1 = w$), the weight of the cable becomes the sole dead load (w_1) in the side spans. Consequently, in reducing Eq. 24 to Eq. 29 for a side-span mode, the multiplying factor w_0/w_1 must be retained. The C term is unchanged, since C, (Eq. 30), contains the factor f/l^2 which is proportional to w_1/w_0. Accordingly the *sole* change in Eqs. 29, 31 and 32 is to replace K_{n_1} for any side-span component by $\frac{w_0}{w_1} K_{n_1}$. This yields:

$$K_1 = \frac{w_0}{w_1} K_{n_1} + C \frac{1}{n_1 a_1} \Sigma' \left(\frac{af}{nl} \right) \tag{33}$$

$$\Sigma' \left[\frac{\frac{1}{n^2} C \frac{f}{l}}{K - \frac{w_0}{w} K_n} \right] = 1 \tag{34}$$

$$\frac{n_1 a_1}{na} = \frac{K - K_n}{K - \frac{w_0}{w_1} K_{n_1}}. \tag{35}$$

Example 6

Straight Backstays

For the hypothetical case of the Mackinac Bridge with straight backstays, assume $L_s = 7540$, $C = 3474$, $C\,f/l = 320.0$.

For the lowest symmetric mode ($n = 1, 3$), Eq. 31 becomes:

$$\frac{320.0}{K - 34.9} + \frac{320.0/9}{K - 440.8} = 1$$

Two solutions:

Dominant	K	Eq. 32
$n = 1$	292.9	$a_3/a = -0.581$
$n = 3$	538.4	$a/a_3 = +0.582$

Compare Example 2. The straight backstays raise the corresponding values of K from 99.5 to 292.9, and from 405.6 to 538.4.

Example 7

Unloaded Backstays (with Cable Sag Considered)

For this case, $\frac{w_1}{w_0} = \frac{2670}{9400} = 0.284$. Assume $l_1 = 1000$ ft. Hence $f_1/l_1 = 0.0069$, $C\,f/l = 320.0$ and $C\,f_1/l_1 = 24.0$.

For the mode: $n = 1, 3$; $n_1 = 1$, (Compare Example 2.)

Eq. 34 becomes: $$\frac{320}{K - 34.9} + \frac{320/9}{K - 440.8} + \frac{2(24.0)}{K - 182.7/0.284} = 1$$

Three solutions:

Dominant	K	Eq. 35
$n = 1$	273.3	$a_3/a = -0.474$
	$(N = 9.24)$	$a_1/a = -0.644$
$n = 3$	496.8	$a/a_3 = 0.364$
	$(N = 12.46)$	$a_1/a_3 = -1.147$
$n_1 = 1$	752.4	$a/a_1 = 0.152$
	$(N = 15.33)$	$a_3/a_1 = 0.117$

Compare Examples 2 and 6. Unloaded backstays with cable sag considered yield K-values somewhat lower than the assumption of "straight backstays," but substantially higher than the K-values with loaded side spans.

COMPACT FORM OF GENERAL EQUATION FOR K

The general formula for K, Eq. 24, may be rewritten in a simple compact expression of equal generality:

$$K = \frac{\Sigma\,(Ka^2l)}{\frac{1}{w_0}\Sigma\,(wa^2l) + \frac{2}{w_0}\Sigma\,(we^2l)} \tag{36}$$

in which K for each span or for each component mode has been computed separately by Eq. 29 or any other pertinent formula.

Equation 36 may also be written directly from the energy relation, Eq. 20, with the aid of Eqs. 2 and 7.

With w equal in all spans and with $e = 0$, Eq. 36 reduces to the simpler form:

$$K = \frac{\sum (Ka^2 l)}{\sum (a^2 l)} \tag{37}$$

which may also be written directly from the energy relation, Eq. 18.

Equation 37 is an exceedingly convenient and useful formula for practical application. It may be used for beams and other structural combinations, as well as for suspension bridges.

OSCILLATIONS WITH LONGITUDINAL MOTION OF SPAN

If $n = 2, 6, 10, \cdots$, without participation of the side spans, the longitudinal motion $\Delta(l/2)$ of the midpoint of the cable may be imparted to the suspended span through any physical coupling at midspan, as when center ties are provided. For this case, by Eqs. 6, 25 and 27,

$$\frac{e}{a} = \frac{16}{\pi} \frac{f}{nl} \tag{38}$$

and Eq. 36 yields:

$$K = \frac{K_0}{1 + 2\left(\frac{16}{\pi} \frac{f}{nl}\right)^2} \tag{39}$$

in which K_0 is given by Eq. 23.

If the two side spans oscillate in opposite phase, with the main span free of vertical oscillation ($a = 0$), Fig. 1*a*, the parallel and synchronous movement Δl_1 of the two tower tops produces longitudinal motion of the main span equal to Δl_1. For this case,

$$\frac{e}{a_1} = \frac{16}{\pi} \frac{f_1}{n_1 l_1} \tag{40}$$

and Eq. 36 yields:

$$K = \frac{K_1}{1 + \frac{l}{l_1}\left(\frac{16}{\pi} \frac{f_1}{n_1 l_1}\right)^2} \tag{41}$$

in which K_1 is given by Eq. 23 for $n_1 = 1, 3, \cdots$.

Equations 39 and 41 express an apparent reduction in K as the equivalent of an increase in the effective value of the oscillating mass m in Eq. 1.

In any segment (l) of a parabolic cable, by Eqs. 25 and 6,

$$\Delta l = \frac{8f}{l^2} \int_0^l \eta dx = \frac{16}{\pi} \frac{af}{nl} \tag{42}$$

For $\frac{l}{2}, \frac{n}{2}, \frac{f}{4}$, $e = \Delta\left(\frac{l}{2}\right)$ and Eq. 42 yields Eq. 38.

For l_1, n_1, f_1, $e = \Delta l_1$, and Eq. 42 yields Eq. 40.

CABLE ANCHORED AT MID-SPAN

If the cable is anchored at midspan, as by center ties, and longitudinal oscillation of the span is prevented (as by friction brakes, or automatically as in the case of torsional oscillations), the antisymmetric modes $n = 2, 6, 10, \cdots$, will induce *side-span participation*, thereby *raising* the value of K. For this case, Eq. 24 or Eq. 29 is applied over the half-span and side span, or from the point of fixation to either anchorage, using the obviously modified values of n, f, and l. (Use $l/2$ for l, $n/2$ for n, $f/4$ for f, $\frac{1}{2}L_s$ for L_s, $2C$ for C, and $2C\ f_1/l_1$ for $C\ f_1/l_1$; a, K_n, and $C\ f/l$ are unchanged.)

Example 8

Antisymmetric Mode with Cable Anchored at Mid-Span

Mode: $n = 2$, $n_1 = 1$ (Fig. 1*b*)

$K_n = 160.7$, 182.7 (Example 1.)

Data: $C\ f/l = 264.0$; $2C\ f_1/l_1 = 249.9$

Applied over the half-span and one side span, Eq. (31) for this case becomes:

$$\frac{264.0}{K - 160.7} + \frac{249.9}{K - 182.7} = 1$$

Two solutions:

Dominant	K	N	Eq. (32)
$n = 2$	171.7	7.32	$a_1/a_2 = -1.000$
$n_1 = 1$	685.5	14.63	$a_2/a_1 = 0.958$

Compare Example 1. The anchorage of the cable at midspan augments K and N. For $n = 2$, by inducing side-span participation, K is increased from 160.7 to 171.7 (increasing N from 7.09 to 7.32 per minute). For $n_1 = 1$, by inducing main-span participation, K is increased from 182.7 to 685.5, augmenting N from 7.55 to 14.63 per minute.

APPROXIMATE METHODS

The last term in the numerator of Eq. 24 represents the effect of ΔH. It is also represented by the C-term in Eq. 29. ΔH is ordinarily either zero (in antisymmetric modes) or of relatively minor magnitude (except for $n = 1$ or $n = 1, 3$). For $\Delta H = 0$ or neglected, K is given by Eq. 23 for a simple mode or by Eq. 36 or 37 for a composite mode.

There will be no cable stretch between anchorages ($\Delta H = 0$) if

$$\Sigma' \left(\frac{af}{nl}\right) = 0 \tag{43}$$

over all spans, or if

$$\Sigma' \left(\frac{a}{n}\right) = 0 \tag{44}$$

in each span—also, of course, for antisymmetric modes.

A simple short-cut for estimating the value of K is to assume that $\Delta H = 0$. Consistent relative amplitudes are then given by Eq. 43 or 44, and their substitution in Eq. 36 or 37 yields K with sufficient accuracy for all practical purposes.

For $n = 3, 5, 7, \cdots$, the governing requirement $\Delta H = 0$, represented by Eq. 44, may be satisfied, without involving the adjoining spans, by superimposing a single-segment oscillation of relative amplitude $-a/n$, yielding, by Eq. 3,

$$\eta = a \sin \frac{\pi n x}{l} - \frac{a}{n} \sin \frac{\pi x}{l}. \tag{45}$$

Substituting K_n and the relative amplitudes in Eq. 37,

$$K = \frac{\pi^2}{l^2}\left(\frac{n^4 + 1}{n^2 + 1}\right) H + \frac{\pi^4}{l^4}\left(\frac{n^6 + 1}{n^2 + 1}\right) EI. \tag{46}$$

Equation 46 may be generalized for any case of superimposed undulations of an even number of different odd values of n. If the same relation is maintained—namely, that the respective amplitudes a of the component harmonics are proportional to their values of n—the generalized form, Eq. 37, reduces to

$$K = \frac{\pi^2}{l^2}\frac{\Sigma n^4}{\Sigma n^2} H + \frac{\pi^4}{l^4}\frac{\Sigma n^6}{\Sigma n^2} EI. \tag{47}$$

For the three spans coupled (to oscillate in unison) as by tower-top movement, with $\Delta H = 0$, Eq. 37 may be written in the form:

$$K = \frac{K_0 + qK_1}{1 + q} \tag{48}$$

in which K_0 and K_1 are the K_n-values by Eq. 23 for the respective spans, and

$$q = \frac{2a_1^2 l_1}{a^2 l}. \tag{49}$$

The main and side spans may oscillate in unison, with a single segment or any *odd* number of segments (n and n_1, respectively) in each span (Fig. 2). If the bridge is symmetrical, the relative amplitudes to satisfy Eq. 43, substituted in Eq. 49, yield

$$q = \frac{n_1^2 l}{2n^2 l_1}. \tag{50}$$

K is then given by Eq. 48.

If $l_1/n_1 = l/n$, the main and side spans are in resonance and $K = K_0 = K_1$.

If $n = 2, 6, 10, \cdots$, then $\sum \Delta(l/n)$ is zero for the full span but is not zero for the half-span. If the midpoint of the cable is not free to take the longitudinal motion $\Delta(l/2)$, as when effectively anchored by center ties, the lengthening and shortening of the respective half-spans will tend to be taken up by side-span participation to produce $\Delta l_1 = \Delta(l/2)$ at each tower top. For this case, Eq. 43 is applied over $l/2$ and l_1. The resulting values of the relative amplitudes, substituted in Eq. 49, yield

$$q = \frac{2n_1^2 l}{n^2 l_1}. \tag{51}$$

K is then given by Eq. 48.

Example 9

Symmetric Mode—Approximate Method by Eqs. 45–47

For the symmetric composite mode $n = 3, 1$, Eq. 46 yields:

$K = 271.7 + 128.5 = 400.2$ ($N = 11.18$). For this case (but with side-span participation, Fig. 2*c*), the more exact method (Example 2) yields $K = 405.6$ ($N = 11.26$). The error of the approximate method is less than 1.5 per cent in the value of K and only 0.7 per cent in the value of N.

Example 10

Antisymmetric Mode—Approximate Method by Eqs. 48–51

For the antisymmetric composite mode $n = 2, n_1 = 1$ (Fig. 1*b*), with center ties forcing side-span participation, Eq. 51 yields $q = 1.056$. With $K_2 = 106.7$ and $K_1 = 182.7$ (from Example 1), Eq. 48 then yields

$$K = 172.0 \qquad (N = 7.33)$$

These results are almost identical with $K = 171.7$, $N = 7.32$, given by the exact method Eq. 31, in Example 8.

SHORT-CUTS FOR ESTIMATING K AND N

Using the cable term alone, Eqs. 23 and 1 yield the convenient approximate relations:

$$K \approx 1.234n^2 \frac{w}{f} \tag{52}$$

and

$$P = \frac{1}{N} \approx \frac{1}{n}\sqrt{f} \tag{53}$$

in which f is simply the sag of the cable. (*Note:* In Eq. 53, P is in seconds and f is in feet.) These simplified formulas, Eqs. 52 and 53, may be used for quickly calculating or estimating the coefficient of rigidity K and the natural period P or frequency N of an unstiffened suspension bridge or of one with negligible stiffening system, for all values of $n > 1$.

Similarly, using the girder term alone, Eqs. 23 and 1 yield the convenient approximate relations:

$$K \approx 1.27n^4 \frac{w}{\eta_w} \tag{54}$$

and

$$P = \frac{1}{N} \approx \frac{1}{n^2}\sqrt{\eta_w} \tag{55}$$

in which η_w is the central deflection theoretically producible (with E constant) by the dead load w acting on the span as a simple beam. (*Note:* In Eq. 55, P is in seconds and η_w is in feet.) Equations 54 and 55 may be used for simple girder spans and for self-anchored suspension spans, for all values of n.

For a stiffened suspension bridge (except the self-anchored type), the composite rigidity K is the sum of the respective contributions (Eqs. 52 and 54) of cable and stiffening girder. Hence, by Eqs. 53 and 55,

$$\frac{1}{P} = N \approx \sqrt{\frac{n^2}{f} + \frac{n^4}{\eta_w}}. \tag{56}$$

Equations 53 and 55 are convenient formulas to remember for quick estimates.

Example 11

K and N by Short-Cut Method

For the Mackinac Bridge:

$w = 9400$ lb./ft. $\quad f = 350$ ft. $\quad \eta_w = 6773$ ft.

For $n = 2$, Eq. 52 yields $K = 132.6$

Eq. 54 yields $K = 28.2$

Total $K = 160.8$

This result is almost identical with the exact value given by Eq. 23. (See Example 1.)

Also, for $n = 2$, Eq. 53 yields $N = 6.41$
Eq. 55 yields $N = 2.92$
Hence, Eq. 56 yields $N = 7.05$/min.

This result is also almost identical with the exact value given by Eqs. 23 and 1. (See Example 1.)

Similar excellent checks of Eqs. 52–56 are obtained for all even values of n.

TORSIONAL OSCILLATIONS

Torsional oscillations may be treated simply as opposed-phase vertical oscillation of the two cables. The angular amplitude at any point of the span is given by

$$\theta = \frac{2\eta}{b} \tag{57}$$

in which b is the width of the bridge measured center to center of girders or trusses.

Ordinarily, the torsional stiffness of the suspended span is relatively negligible. Hence, unless substantial torsional stiffness is provided by special construction (as with a double system of lateral bracing), the values of K and N determined for vertical oscillations, multiplied by simple dimensional constants, yield with sufficient accuracy the corresponding values for torsional oscillations. The spring constant of the span in torsion, or the coefficient of torsional rigidity, will be

$$K_t = \left(\frac{b}{2}\right)^2 K = (mr^2)\omega_t^2 \tag{58}$$

and the natural frequency will be

$$N_t = \frac{b}{2r} N \tag{59}$$

$$2\pi N_t = \omega_t = \frac{b}{2r}\omega \tag{60}$$

in which r is the polar radius of gyration of the mass of the cross section of the span. It is important to note that, for torsional resistance of a suspension bridge, the polar moment of inertia (r^2m) pertains, not to a localized built-up cross section at the panel points, but to the average cross section throughout, including the long ranges between panel points. For the usual proportions, $b/(2r) = 1.3$ to 1.5.

The polar radius of gyration r is given by

$$r^2 = \frac{\Sigma\,(mr^2)}{\Sigma\,m} \tag{61}$$

in which the component r for the cable masses is $b/2$, but for all elements of the suspended section it is the *polar* radius to each element from the *center of oscillation* of the section.

TORSIONAL RESISTANCE OF WIND TRUSSES

In the derivation of the foregoing formulas for K, the effect of web shear in the stiffening truss or girder has been neglected. The effect of the web shear contribution is found to produce a comparatively infinitesimal or negligible reduction in K. The normal omission of the web shear contribution from the K formulas is completely justified.

If, however, wind trusses are provided in the horizontal planes of the top and bottom chords (or flanges) of the stiffening trusses (or girders), new relations between the bending and shear components come into play in *torsional* displacements. The interaction of the horizontal and vertical trusses (or girders), with common chords, imposes a relative reversal of sign in the bending components of the horizontal trusses, with a consequent enforced increase in all of the shear components in the deflection and energy equations. In such case the shear terms are no longer negligible but contribute a valuable *increase* in K.

Accordingly, a highly effective method of augmenting resistance to *torsional* oscillations is by providing two planes of lateral bracing, at or near the top and bottom chords (or flanges) of the stiffening trusses (or girders), respectively, so as to secure the integral effect of a hollow rectangular section in torsion. The cross section is assumed rigidly constructed or braced so as to preserve its rectangular form. The resultant transverse *shearing* forces in the four planes of the tubular member, when it is subjected to torsion, will then contribute very material increases in K_t and R.

Let η and z represent the vertical and lateral displacements, respectively, at any section of the chord (or flange). Each is the algebraic sum of bending and shear contributions:

$$\eta = \eta_m + \eta_s \qquad z = z_m + z_s. \tag{62}$$

Between the corresponding elements in the vertical and horizontal planes (distinguished by subscripts where necessary), the five governing static conditions of equal rotation, equal chord stress, identical chords, shear equilibrium, and ratio of shear deflections, in the order named, yield the following five simple basic relations:

$$\frac{\eta}{z} = \frac{b}{d}, \quad \frac{\eta_m}{z_m} = -\frac{b}{d}, \quad \frac{I_v}{I_h} = \frac{d^2}{b^2}, \quad \frac{S_v}{S_h} = \frac{d}{b}, \quad \frac{\eta_s}{z_s} = k\frac{b}{d} \tag{63}$$

in which d is the vertical truss depth and

$$k = \frac{A_h d^2}{A_v b^2} \tag{64}$$

(non-dimensional). A_v is the vertical web section of the girder, multiplied by E_s/E; or, in the case of a truss, A_v is the equivalent solid web section and is given by

$$A_v = A_d (\cos^2 \phi \sin \phi) \tag{65}$$

in which A_d is the sectional area of the diagonal member or members in a truss panel, ϕ is the inclination of the diagonal from the vertical, and E_s is the shear modulus of elasticity. (For steel, E_s/E may be taken as 0.385.) A_h is similarly defined in each horizontal plane of bracing. In Eqs. 63, note the enforced reversal of sign for the z_m ratio; this is significant.

From Eqs. 62 and 63, the component deflections η_m, η_s, z_m, z_s and z are easily written as simple ratios of η, containing only b, d, and k. Also, substituting η_s in the shear geometry of a girder web yields

$$S_v = \frac{2k}{1+k} EA_v \frac{d\eta}{dx}. \tag{66}$$

In the case of a truss, the geometry of a truss panel, with the aid of Eq. 65, yields the identical formula, Eq. 66.

Substitution of the foregoing relations, Eqs. 62–66, in the energy equation, as in Eqs. 16–22, yields the desired K-formula:

$$K = \frac{\pi^2 n^2}{l^2} H + \left(\frac{1-k}{1+k}\right)^2 \frac{\pi^4 n^4}{l^4} EI + \left(\frac{4k}{1+k}\right) \frac{\pi^2 n^2}{l^2} EA_v \tag{67}$$

in which the three terms represent the respective contributions of cable displacement, chord (or flange) stresses, and diagonal (or shear) stresses. This last term is new, and constitutes a very significant addition to K and an even more significant addition to R.

In Eq. 67, the moment contributions of the vertical and the horizontal trusses are equal; but the corresponding coefficient 2 is omitted from the second term of Eq. 67, because the equality in this case represents a duplication of identical chords counted in both the horizontal and vertical trusses.

Without the two planes of horizontal trussing, $A_h = 0$, $k = 0$, and Eq. 67 reduces to Eq. 23.

Equation 67 replaces K_n in previous formulas. If ΔH is not zero, add the ΔH term as in Eq. 29. For simple antisymmetric modes ($n = 2, 4, 6, \cdots$), Eq. 67 is complete.

K should preferably be written to represent the complete cross section of the bridge, rather than the value for one side or one cable. Accordingly, in Eq. 67, H refers to the *two* cables, I and A_v refer to the *two* vertical trusses or girders, and A_h refers to the *two* horizontal trusses.

For the usual case of *two* cables and stiffening trusses, the *total* value of K multiplied by $b^2/4$ gives K_t (in torsion), as in Eq. 58.

Equation 67 assumes the two vertical stiffening trusses (and cables) oscillating in *opposite* phase. Consequently Eq. 67 does not apply to ordinary vertical oscillations, and does *not* represent *vertical* rigidity of the structure.

By Eq. 67, the addition of wind trusses (in two horizontal planes) is a highly economical and effective method of providing desired resistance to *torsional* oscillations. It is vastly surpassed, however, in economy and efficiency by the simple device of installing transverse diagonal stays, between the cables and the trusses, as described and successfully applied by the author in 1943.

Example 12

Antisymmetric Mode ($n = 2$) in Torsional Oscillations

For $n = 2$, Example 1, by Eq. 23, gives

$$K = 160.7 \qquad (N = 7.09)$$

Hence, by Eqs. 57–61, with $b = 68$ ft., $r = 30.4$ ft.

$$K_t = 185{,}800 \qquad (N_t = 7.93/\text{min.})$$

With center ties, causing side-span participation (Fig. 1*b*), Example 8, by Eq. 31 applied over the half-span and one side span, gives

$$K = 171.7 \qquad (N = 7.32).$$

Hence, by Eqs. 57–61,

$$K_t = 198{,}500 \qquad (N_t = 8.19).$$

For the case with two horizontal planes of lateral bracing, we use the data of Example 1 together with $d = 38$, $b = 68$, $A_v = 15.62$, $A_h = 23.60$, $k = 0.472$. With these data, Eq. 67 gives:

$$K = 132.5 + 3.6 + 1588.4 = 1724.5.$$

Hence, by Eqs. 57–61,

$$K_t = 1{,}993{,}500 \qquad (N_t = 25.96).$$

Compared with $K_t = 185{,}800$, this represents a 973 per cent increase in K_t (and a 227 per cent increase in N_t) due to the provision of the double system of lateral bracing.

With center ties, causing side-span participation (Fig. 1*b*), in addition to the double system of lateral bracing, Eq. 31 applied over the half-span

and one side span, gives:

$$\frac{264.0}{K - 1724.5} + \frac{2(125.0)}{K - 1922.0} = 1$$

This yields, for $n = 2$ dominant,

$K = 1807.5$ (and $K = 2353.0$ for $n_1 = 1$ dominant).

Hence, by Eqs. 57–61,

$K_t = 2{,}089{,}500$ ($N_t = 26.58$).

Compared with the corresponding value of $K_t = 198{,}500$, this represents a 953 per cent increase in K_t (and a 225 per cent increase in N_t) due to the provision of a double system of lateral bracing.

Example 13

Symmetric Mode ($n = 1$) in Torsional Oscillations

For the composite mode $n = 1, n_1 = 1$, Fig. 2*a*, Eq. 31 yields $K = 100.3$ ($N = 5.60$). Hence, by Eqs. 57–61,

$K_t = 115{,}900$ ($N_t = 6.26$).

For the composite mode $n = 1, 3$, with $n_1 = 1$, Fig. 2*b*, Example 2, by Eq. 31, gives $K = 99.5$ ($N = 5.57$). Hence, by Eqs. 57–61,

$K_t = 115{,}000$ ($N_t = 6.23$).

This value is slightly lower than $K_t = 115{,}900$, and represents the desired minimum.

For the case with two horizontal planes of lateral bracing, with the data of Example 12, Eq. 67 gives:

$$K_n = 33.1 + 0.2 + 397.1 = 430.4$$

for $n = 1$, also $K_n = 3890.4$ for $n = 3$, and $K_n = 1922.0$ for $n_1 = 1$. With these values of K_n, for the composite mode $n = 1, 3$, with $n_1 = 1$, Fig. 2*b*, Eq. 31 becomes:

$$\frac{264.0}{K - 430.4} + \frac{264.0/9}{K - 3890.4} + \frac{2(125.0)}{K - 1922.0} = 1.$$

This yields, for $n = 1$ dominant, $K = 649.4$. (The relative amplitudes are given by Eq. 32.) Hence, by Eqs. 57–61,

$K_t = 750{,}700$ ($N_t = 15.93$).

Compared with $K_t = 115{,}000$, this represents a 553 per cent increase in K_t (and a 156 per cent increase in N_t) due to the provision of a double system of lateral bracing.

For the composite mode $n = 1$, $n_1 = 1$, Fig. 2*a*, Eq. 31 takes the simpler form:

$$\frac{264.0}{K - 430.4} + \frac{2(125.0)}{K - 1922.0} = 1.$$

This yields, for $n = 1$ dominant, $K = 651.0$. Hence, by Eqs. 57–61,

$$K_t = 752{,}600 \qquad (N_t = 15.95).$$

The effect of omitting the component $n = 3$ thus appears to be practically negligible. The lower value ($K_t = 750{,}700$) represents the desired minimum.

EFFECT OF TORSIONAL STIFFNESS OF THE TOWERS

Suspension bridge towers are normally designed as flexible cantilevers, fixed at the base. Their contribution to K, the stiffness of the bridge against *vertical* oscillations, is small and usually negligible.

The *torsional* stiffness of the towers, however, offers a larger contribution to K_t, the resistance of the bridge to *torsional* oscillations. This contribution may be material, even without special design of the towers for greater torsional stiffness. In comparison with other methods, such special design of the towers would usually be an uneconomical means of increasing the torsional rigidity of the bridge.

A couple formed by a horizontal cable force P at each saddle will twist the tower about its vertical axis to produce a longitudinal displacement e at each saddle. The torsional stiffness of the tower is denoted by

$$K_T = \frac{P}{e}. \tag{68}$$

Let

$L_{s0} = L_s$ for $\frac{1}{2}$ main span.

$L_{s1} = L_s$ from anchorage to tower.

$$(L_{s0} + L_{s1} = \tfrac{1}{2}L_s)$$

$\Delta H_1 = \Delta H$ from anchorage to tower.

$\Delta H_0 = \Delta H_1 + P = \Delta H$ in $\frac{1}{2}$ main span.

Considering the main span,

$$e = \frac{8}{\pi}A - \frac{\Delta H_0 L_{s0}}{E_c A_c}. \tag{69}$$

Considering the side span,

$$e = -\frac{8}{\pi}B + \frac{\Delta H_1 L_{s1}}{E_c A_c} \tag{70}$$

in which

$$A = \frac{\pi f}{l^2}\int_0^{l/2} \eta dx = -\sum \frac{af}{nl}\left(\cos\frac{n\pi}{2} - 1\right)$$

$$B = \frac{\pi f_1}{l_1^2}\int_0^{l_1} \eta_1 dx = -\sum \frac{a_1 f_1}{n_1 l_1}(\cos n_1\pi - 1). \tag{71}$$

Equating the two expressions for e, the foregoing equations yield:

$$\Delta H_1 = -\frac{16}{\pi}\frac{1}{U}[-(E_cA_c)A + (E_cA_c + K_TL_{s0})B]$$

$$\Delta H_1 + P = -\frac{16}{\pi}\frac{1}{U}[(E_cA_c)B - (E_cA_c + K_TL_{s1})A] \qquad (72)$$

in which

$$U = L_s + \frac{2K_tL_{s0}L_{s1}}{E_cA_c}. \qquad (73)$$

The total potential energy due to cable stretch (ΔH) in the whole bridge and including the torsion in the *two* towers will then be:

$$W_p = \frac{256}{\pi^2 U}\left[E_cA_c(A+B)^2 + 2K_TL_{s1}A^2 + 2K_TL_{s0}B^2 + \frac{4K_T}{U}(L_{s1}A - L_{s0}B)^2\right]. \qquad (74)$$

The first three terms in the bracket represent W_p due to cable stretch, and the fourth term represents W_p due to torsion in the two towers.

For $K_T = 0$, Eq. 74 reduces to the ΔH term in Eq. 24.

The potential energy due to truss distortion has already been determined, as represented in Eq. 67, and gives the first term of the numerator of the following equation. The kinetic energy of the towers is negligible. From Eq. 74 and $W_p = W_k = \frac{1}{2}K\sum a^2l$, we obtain the equation for K:

$$K = \frac{\sum(K_na^2l) + \frac{512}{\pi^2U^2}[E_cA_cL_s(A+B)^2 + 2K_T(L_s+U)(L_{s1}A^2 + L_{s0}B^2)]}{\sum a^2l} \qquad (75)$$

in which K_n is given by Eq. 23 or 67 for each component mode.

Equation 75 gives the numerical value of K in terms of the relative amplitudes (a_1/a). For a composite mode consisting of one main-span mode (amplitude a) and one side-span mode (amplitude a_1), the partial derivative $\frac{\partial K}{\partial a} = 0$ yields K and a_1/a for the mode n dominant; and the partial derivative $\frac{\partial K}{\partial a_1} = 0$ yields K and a/a_1 for the mode n_1 dominant. These values will be virtually unchanged if an additional mode of higher n is superimposed. For a composite mode consisting of three or more simple modes superimposed, the resulting simultaneous equations may be solved graphically.

In applying Eq. 75, K_n and A_c should be taken as the values for the complete cross-section of the bridge (*two* cables and *two* vertical trusses or girders), in order that Eq. 58 may then be applied to the resulting value of K to give K_t.

From K given by Eq. 75, the values of K_t and N_t are then given by Eqs. 58 and 59.

Equation 75 assumes the two vertical stiffening trusses and cables oscillating in *opposite* phase. Consequently Eq. 75 does not apply to ordinary vertical oscillations, and does *not* represent *vertical* rigidity of the structure.

Example 14

Antisymmetric Mode with Torsional Stiffness of Towers

(Fig. 1*b*)

For the Mackinac Bridge:

$L_s = 9140$ ft. $\quad E_cA_c = 20.82\ (10)^9$ lb.

$L_{s0} = 2029$ ft. $\quad K_T = 873{,}000$ lb./ft.

$L_{s1} = 2541$ ft. $\quad U = 10{,}005$ ft.

$H = 48{,}478{,}000$ lb. $\quad f/l = 0.0921 \quad f_1/l_1 = 0.0436$

By Eq. 67, $K_n = 1724.5$ for $n = 2$ and $K_n = 1922$ for $n_1 = 1$

(Compare Example 12.)

For the antisymmetric mode, $n = 2$, $n_1 = 1$, Eq. 75 then yields

$$K = \frac{7.764a^2 + 1.584a_1a + 7.936a_1^2}{0.0038a^2 + 0.0036a_1^2}.$$

For $n = 2$, dominant, $\dfrac{\partial K}{\partial a} = 0$ yields:

$$K = 1895, \quad \text{and} \quad a_1/a = -0.711.$$

Hence

$$K_t = 2{,}190{,}600 \qquad (N_t = 27.21).$$

For $n_1 = 1$, dominant, $\dfrac{\partial K}{\partial a_1} = 0$ yields:

$$K = 2352.8, \quad \text{and} \quad a/a_1 = 0.674.$$

Hence

$$K_t = 2{,}719{,}800 \qquad (N_t = 30.32).$$

Comparison with the corresponding values in Example 12 shows that the effect of including the torsional stiffness of the towers is an increase of 4.8 per cent in K_t (and 2.4 per cent in N_t) for the mode $n = 2$, dominant.

Example 15

Symmetric Mode with Torsional Stiffness of Towers
(Fig. 2*a*)

Data as in Example 14.

By Eq. 67, $K_n = 430.4$ for $n = 1$,

and $K_n = 1922.0$ for $n_1 = 1$.

(Compare Example 13.)

For the symmetric mode $n = 1$, $n_1 = 1$, Eq. 75 then yields

$$K = \frac{2.846a^2 + 1.584a_1a + 7.936a_1^2}{0.0038a^2 + 0.0036a_1^2}.$$

For $n = 1$, dominant, $\frac{\partial K}{\partial a} = 0$ yields:

$$K = 718.2 \quad \text{and} \quad a_1/a = -0.148$$

Hence

$$K_t = 830{,}200 \qquad (N_t = 16.75).$$

For $n_1 = 1$, dominant, $\frac{\partial K}{\partial a_1} = 0$ yields:

$$K = 2235.2 \quad \text{and} \quad a/a_1 = 0.140$$

Hence

$$K_t = 2{,}583{,}900 \qquad (N_t = 29.55).$$

Comparison with the corresponding values in Example 13 shows that the effect of including torsional stiffness of the towers is an increase of 10.3 per cent in K_t (and 5.0 per cent in N_t) for the mode $n = 1$, dominant.

If we superimpose the harmonic $n = 3$ on the oscillation $n = 1$, $n_1 = 1$, the numerical results obtained above are very slightly modified. The value of K_t for $n = 1$, dominant, is reduced from 830,200 to 826,800; and the value of K_t for $n_1 = 1$, dominant, is reduced from 2,583,900 to 2,579,000. The reduction in K_t and N_t is insignificant, and such refinement is not justified.

EFFECT OF CENTER OF SUSPENSION

The effective center of suspension of a suspension bridge cross section in torsional oscillation is the midpoint of the line joining the two points of connection of the suspenders to the outside girders or trusses. The torsional stability of the section is increased by raising this center of suspension relative to the center of gravity of the suspended section. This is confirmed by simple model tests. The spring constant K_t is augmented by a torque of gravity restitution; a gravity pendulum effect is added to the torsional spring constant.

If h is the vertical height of the center of suspension above the center of gravity of the suspended section, the vertical rise of the center of

gravity of the suspended section, as it swings in an arc like a pendulum, is, by Eq. 57,

$$v = h\frac{\theta^2}{2} = \frac{2h}{b^2}\eta^2 \tag{76}$$

and the potential energy is augmented by

$$\Delta W_p = w_s \cdot v \tag{77}$$

in which w_s is the suspended dead load (exclusive of the cables) per unit length. Accordingly, by Eqs. 16–20, and 57–61, the value of K_t given by Eq. 58 is augmented to

$$K_t = \left(\frac{b}{2}\right)^2 K + w_s h. \tag{78}$$

The total energy W required for any angular amplitude is increased in the same proportion, and consequently the aerodynamic instability, measured by the rate of amplification, is reduced in the same ratio.

In computing the polar radius of gyration r by Eq. 61, the *center of oscillation* of the suspended section was specified. If the polar moment of inertia mr^2 has been computed about the center of gravity of the suspended section, a correction must be made when the center of oscillation is at a height h above the center of gravity. The polar moment of inertia is augmented by

$$\Delta(mr^2) = m_s h^2. \tag{79}$$

By combining Eqs. 78 and 79 we may write the equivalent value of K_t:

$$K_t = mr^2\omega_t^2 = \frac{\left(\frac{b}{2}\right)^2 K + w_s h}{1 + \frac{w_s}{w}\left(\frac{h}{r}\right)^2}. \tag{80}$$

In Eq. 80, the numerator represents the augmented potential energy and the denominator represents the augmented kinetic energy. The kinetic energy is augmented by the horizontal velocity of the center of gravity along the arc of the pendulum swing.

Instead of increasing h by raising the points of suspender connection, as by deeper girders or by brackets, the same improvement may be accomplished by lowering the center of gravity of the suspended section. Accordingly, torsional rigidity and stability may be improved by changing from deck or half-through construction to through construction. The author has demonstrated this conclusion by model tests. An unstable deck truss section became aerodynamically stable when simply inverted.

Example 16

Effect of Center of Suspension
For the Mackinac Bridge:

$w = 9400$, $b = 68$ ft.
$r = 30.4$ ft.
$w_s = 6530$, $h = 6.96$ ft.

Hence Eq. 80 becomes:

$$K_t = \frac{(b/2)^2 K + 45{,}449}{1.0364}.$$

For $n = 2$ (See Example 12), K_t is thus increased from 185,800 to 223,100, an increase of 20 per cent.

For $n = 1$ (See Example 13), K_t is thus increased from 115,900 to 155,700, an increase of 34 per cent.

For the much higher values of K_t, with two horizontal planes of lateral bracing, the effect of the raised center of suspension ($h = 6.96$) is found to be negligible.

VARIATIONAL PRINCIPLE FOR TRANSPORT PHENOMENA

BY

G. E. TAUBER[1]

SUMMARY

The role of the variational principle in transport theory and its application to the calculation of the transport coefficients are described. A generalized variational principle including the effect of an external magnetic field and lattice vibrations is discussed. The variational principle for inter-band scattering is derived, and is generalized by the introduction of an external magnetic field and lattice vibrations, leading to the most general variational principle including inter-band scattering, in addition to an external magnetic field and lattice vibrations, or any combination of these. The transport coefficients are expressed in terms of energy integrals, which for the various cases take the form of ratios of infinite determinants, the number and form of their matrix elements depending on the interactions. A formal solution is given by the Ritz method for the different cases of interest.

I. INTRODUCTION

Many physical problems can be formulated in terms of a variational principle, the solution of which is often simpler and more accurate than the original formulation. This is due to the fact that the variational function is not as sensitive to fluctuations and errors as the differential (or integral) equation obtained upon variation. For practical calculations use is made of the Ritz variational method in which the unknown function is expanded as an infinite series with coefficients determined by the variational method (1).[2]

In the case of transport phenomena the fundamental equation determining the distribution function, and thus the various transport phenomena, is an integro-differential equation known as the Boltzmann equation (2). We can write this equation as an operator equation of the form

$$L(F) = X \tag{1}$$

where L represents the integral operator due to the change of the distribution function by collisions, and X the force due to the external field and temperature gradients. It has been shown by Kohler (3) that this equation is equivalent to a variational principle, for which the variational function is simply the inner product

$$(F; F) = \int FL(F)d\mathbf{k} \tag{2}$$

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[2] The boldface numbers in parentheses refer to the references appended to this paper.

subject to the subsidiary condition

$$(F, X) = \int FXd\mathbf{k} = (F; F) \text{ on account of (1).} \tag{3}$$

It has been pointed out by Ziman (4) that the variational function $(F; F)$ is not simply a mathematical device, but of physical interest as it is proportional to the rate of entropy production by collision processes, and thus affords a connection between microscopic statistical theory and irreversible thermodynamics. The positive definiteness of that function based on microscopic reversibility of the operator L assures us also that the Onsager reciprocity relations (5) among the various transport quantities are always satisfied (6).

If we now expand the function F in a series of the form

$$F = \sum_r a_r \epsilon^r \tag{4}$$

where ϵ is the energy, variation with respect to the coefficients a_r will result in a system of linear equations for these quantities

$$\alpha_r = \sum_s d_{rs} a_s \tag{5}$$

where the coefficients α_r and d_{rs} can be evaluated from a knowledge of the form of L and X. The various transport coefficients, such as electrical and heat conductivity, can then be obtained in terms of energy integrals K which turn out to be just the subsidiary conditions (3). It has been shown by Enskog (7) that these energy integrals can be expressed in terms of infinite determinants, the elements of which can be evaluated for a given energy-momentum dependence.

If deviations from equilibrium of the phonon distribution function are also taken into account this results in a system of coupled integral equations, which in operator form are given by

$$\begin{aligned} X &= L_1(F) + L_2(N) \\ Y &= L_3(F) + L_4(N). \end{aligned} \tag{6}$$

Here N represents the phonon distribution function, in addition to the previously discussed carrier distribution function F. Furthermore, in addition to the operator $L_1 = L$ (of 1) there also appear the operators L_2, L_3 and L_4 describing the interaction of the electrons with the phonons and phonons with phonons. A variational approach to this problem has been suggested by Ziman (4), and recently Dorn (8) obtained a variational principle and applied it to include the effect of longitudinal polarized vibrations for spherical energy surfaces. The

variational function is given by the sum of terms of the form (2)

$$(F;F)_1 + (F;N)_2 + (N;F)_3 + (N;N)_4 \tag{7}$$

where the subscripts $i = 1$ to 4 refer to the appropriate operator L_i. The symmetry relations amongst the integral operators assure us of the positive definiteness of the variational function (7) and the existence of the Kelvin relations amongst the transport coefficients. If the lattice distribution function N is also expanded in terms of an infinite series analogous to the corresponding one for F (4) an application of the Ritz variational method will then again result in a system of linear equations for the unknown coefficients. The energy integrals are again given in terms of infinite determinants, the elements of which now depend on a knowledge of the operators L_i ($i = 1$ to 4) and the functions X and Y.

In the presence of an external magnetic field the above theory has to be modified. The Boltzmann equation for the carrier distribution function acquires an extra term, which represents the effect of the "magnetic scattering." Taking the symmetry property of the magnetic scattering operator under reversal of the field into account, it is still possible to obtain the Kelvin relations for that case (9). Although the magnetic field does not contribute to the entropy production it is possible to formulate a variational principle (10) which leads to the correct transport equation. Its solution is again achieved by an application of the Ritz method, except that two separate expansions for the conjugate sets F^+ and F^- must be used, depending on the direction of the magnetic field $\mathbf{H}$ and $-\mathbf{H}$.

We have shown (11) how a generalized variational principle can be found, which is valid for an arbitrary direction of the electric field and polarization of the lattice vibrations in the presence of an external magnetic field, and does not depend on any special form of the energy surface. The various transport coefficients, both for thermoelectric and thermomagnetic phenomena, are then obtained by the Ritz method in terms of infinite determinants without requiring an explicit solution of the transport equation.

In some cases of physical interest, notably in semiconductors, one has more than one energy surface, and it becomes necessary to consider the effect of transitions between bands. The appropriate operator equation is then given by

$$\sum_n L_{mn}(F^n) = X_m \tag{8}$$

where the subscripts m, n refer to the various bands. The variational function for this case turns out to be a sum of terms of the type given earlier (2)

$$(F;F) = \sum_m \sum_n (F^m;F^n) \tag{9}$$

subject to the subsidiary conditions

$$(F, X) = \sum_m (F^m, X_m) = (F; F) \text{ on account of (8).} \qquad (10)$$

This variational principle can again be solved by an application of the Ritz method where the coefficients a_r (of 4) now acquire an extra subscript indicating the band m and become a_{mr}. Carrying out the variation with respect to these parameters one again obtains a set of linear simultaneous equations for these unknowns of the form (5), the difference being the appearance of extra indices in the quantities α_r and d_{rs} indicating the bands. The current density $\mathbf{J}$ and heat current density $\mathbf{W}$ may be considered as a sum over bands over the appropriate expressions for the individual bands

$$\mathbf{J} = \Sigma \mathbf{J}_m \qquad \text{and} \qquad \mathbf{W} = \Sigma \mathbf{W}_m. \qquad (11)$$

The transport coefficients are again expressed in terms of energy integrals K which can be obtained from infinite determinants without the necessity of explicitly solving the system of equations for a_{mr}, the only difference being that now there appear more matrix elements including the contributions from all different bands m, n.

A further generalization of the theory can be achieved by considering the effect of an external magnetic field or/and the presence of lattice vibrations for this case. If the non-equilibrium phonon distribution function is taken into account a system of coupled equations of the form (6) is obtained

$$\begin{aligned} X_m &= \sum_n L_{mn}^{1}(F^n) + \sum_n L_{mn}^{2}(N) \\ Y &= \sum_{m,n} L_{mn}^{3}(F^n) + \sum_{m,n} L_{mn}^{4}(N) \end{aligned} \qquad (12)$$

and a variational principle can again be formulated, the variational function being of the type (7) where each term is now a sum over bands of the type (9). The effect of an external magnetic field again introduces an extra scattering operator L' into the formalism and produces a separation of the solution into two conjugate sets depending on the direction of the magnetic field $\mathbf{H}$. The energy integrals, and hence the transport coefficients, can again be obtained by the Ritz method and expressed in terms of infinite determinants, containing now many more matrix elements describing the effect of the interaction of the electrons with the lattice, phonon-phonon interaction, and the various bands.

The purpose of this paper is to give as complete a discussion as is possible of the variational principle in transport theory and its application to the calculations of the various transport coefficients. Starting

with the derivation of the variational principle developed by Kohler[3] we include the effect of an external magnetic field or lattice vibrations, and finally combine both these effects in a discussion of the generalized variational principle. The variational principle for inter-band scattering is set up next, and is afterwards generalized by the introduction of an external magnetic field and lattice vibrations, so that we finally obtain a variational principle which includes inter-band scattering, in addition to an external magnetic field and lattice vibrations. In order to present a systematic presentation, Section II is devoted to the Boltzmann equation and contains a discussion of the various cases, *viz.* effect of magnetic field, lattice vibrations and inter-band scattering, and a combination of any or all of these. In Section III, the appropriate variational principles are derived and their solution indicated by an application of the Ritz method. Finally, in Section IV, starting from a general expression of the transport coefficients in terms of energy integrals, it is shown how these can be expressed in terms of infinite determinants, which differ from each other in the number and nature of their matrix elements. In order to evaluate these matrix elements, the form of the integral operators must be known, together with the dependence of energy on momentum. Work along these lines is in progress, and has been reported elsewhere (12), while here we have only been concerned with presenting a systematic discussion of the variational principle applied to transport phenomena and in particular, expressing the various energy integrals necessary for the calculation of the transport coefficients, in terms of infinite determinants for the various cases of interest.

II. BOLTZMAN TRANSPORT EQUATION

1. *Nature of Transport Equation*

For a good analysis of transport phenomena it is necessary to determine the distribution of the particles both in regard to position and velocities. The fundamental equation determining the distribution function $f(\mathbf{v}, \mathbf{r})$ is an integro-differential equation known as the Boltzmann equation, which describes the probability of a particle being in a given state, and is therefore based on the one-electron model. It is, however, quite simple to generalize the distribution function, and hence its change with time to a n-dimensional phase space in which it is now a function of the co-ordinates and momenta of the n particles. Recently, several authors have adopted the approach of the density matrix in quantum statistics to derive the transport equation from general statistical consideration. However, for the present presentation, we will assume the validity of the standard theory and postpone a more rigorous derivation of the transport equation to a future discussion.

[3] *Loc. cit.*

If the number of particles which lie in an element of $d\mathbf{r}$ of configuration space and in $d\mathbf{k}$ of $\mathbf{k}$-space at a time t is given by

$$(4\pi^3)^{-1} f(\mathbf{k}, \mathbf{r}, t) d\mathbf{k} d\mathbf{r} \tag{12}$$

then at time $t + dt$ according to Liouville's theorem they will lie in an equal volume of six dimensional phase space about $\mathbf{k} + \dot{\mathbf{k}}dt$, $\mathbf{r} + \mathbf{v}dt$. The difference, therefore, between $f(\mathbf{k} + \dot{\mathbf{k}}dt, \mathbf{r} + \mathbf{v}dt, t + dt)$ and $f(\mathbf{k}, \mathbf{r}, t)$ must be due to the particles scattered into the new positions in the time dt. The total change of the distribution function f due to collisions with the lattice (and each other) and external fields must be zero, so that

$$(\partial f/\partial t)_{\text{coll.}} + (\partial f/\partial t)_{\text{fields}} = 0 \tag{13}$$

where $(\partial f/\partial t)_{\text{coll.}}$ represents the change due to collisions and $(\partial f/\partial t)_{\text{fields}}$ the change due to external fields, which is given by

$$df/dt = \partial f/\partial t + \dot{\mathbf{k}} \cdot \nabla_{\mathbf{k}} f + \mathbf{v} \cdot \nabla_{\mathbf{r}} f. \tag{14}$$

Here $\hbar\dot{\mathbf{k}}$ is the change of the impulse and equal to the external force

$$\hbar\dot{\mathbf{k}} = e\mathbf{E} + e(\mathbf{v}/c) \times \mathbf{H} \tag{15}$$

where $\mathbf{E}$ denotes the electric field, $\mathbf{H}$ the magnetic field, e the charge and c the velocity of light. The velocity of the electron $\mathbf{v}$ is expressed in terms of its wave vector $\mathbf{k}$ through

$$\hbar\mathbf{v} = \nabla_{\mathbf{k}} \epsilon(k_x, k_y, k_z). \tag{16}$$

(In the steady state one also has $\partial f/\partial t = 0$.)

a. Collision Operator

Since binary collisions of electrons are neglected, the number of particles per unit volume with velocities in the range $d\mathbf{v}$ which have their velocities changed to lie in $d\mathbf{v}'$ in time dt must be proportional to $f(\mathbf{v}, \mathbf{r}, t)$ and to the interaction $W(\mathbf{v}, \mathbf{v}', \mathbf{r})$, which is independent of f and only depends on the interaction mechanism. Similarly, the number of particles which are scattered into the range dv are proportional to $f(\mathbf{v}', \mathbf{r}, t)$ and the interaction $W(\mathbf{v}', \mathbf{v}, \mathbf{r})$. The net difference between these two quantities, integrated over $d\mathbf{v}'$, determines $(\partial f/\partial t)_{\text{coll.}}$, so that

$$(\partial f/\partial t)_{\text{coll.}} = f(\mathbf{v}', \mathbf{r}, t) W(\mathbf{v}', \mathbf{v}, \mathbf{r}) - f(\mathbf{v}, \mathbf{r}, t) W(\mathbf{v}, \mathbf{v}', t) d\mathbf{v}'. \tag{17}$$

The equilibrium distribution function f_0 must be such that $(\partial f_0/\partial t)_{\text{coll.}} = 0$, and thus

$$W(\mathbf{v}, \mathbf{v}', \mathbf{r})/W(\mathbf{v}', \mathbf{v}, \mathbf{r}) = f_0(\mathbf{v}', \mathbf{r})/f_0(\mathbf{v}, \mathbf{r}).$$

Now according to classical mechanics, f_0 is a function of the energy only, and in fact, is the Maxwell-Boltzmann function. Also, the collisions are supposed to be elastic, so that energy is conserved and $f_0(\mathbf{v}, \mathbf{r}) = f_0(\mathbf{v}', \mathbf{r})$. Hence $W(\mathbf{v}, \mathbf{v}', \mathbf{r}) = W(\mathbf{v}', \mathbf{v}, \mathbf{r})$ and

$$(\partial f/\partial t)_{\text{coll.}} = \int [f(\mathbf{v}', \mathbf{r}, t) - f(\mathbf{v}, \mathbf{r}, t)]W(\mathbf{v}, \mathbf{v}', \mathbf{r})d\mathbf{v}'. \quad (18)$$

In the case of electrons, Pauli's exclusion principle has to be taken into account. We describe the states of the electrons by the wave vector $\mathbf{k}$ (rather than the velocity $\mathbf{v}$) and suppose the $W(\mathbf{k}, \mathbf{k}')$ is the probability that an electron makes a transition from the state $\mathbf{k}$ to the state $\mathbf{k}'$. As a transition can only occur if the corresponding state is unoccupied, the number of electrons, per unit volume, gained by $d\mathbf{k}$ from $d\mathbf{k}'$ in time dt is

$$W(\mathbf{k}', \mathbf{k})f(\mathbf{k}')[1 - f(\mathbf{k})]d\mathbf{k}d\mathbf{k}'dt$$

and the number of electrons lost from $d\mathbf{k}$ to $d\mathbf{k}'$ is

$$W(\mathbf{k}, \mathbf{k}')f(\mathbf{k})[1 - f(\mathbf{k}')]d\mathbf{k}d\mathbf{k}'dt$$

so that the net change in f is given by

$$(\partial f/\partial t)_{\text{coll.}} = \int W(\mathbf{k}', \mathbf{k})f(\mathbf{k}')[1 - f(\mathbf{k})] - W(\mathbf{k}, \mathbf{k}')f(\mathbf{k})[1 - f(\mathbf{k}')]d\mathbf{k}'. \quad (19)$$

This expression must vanish in the equilibrium state, where there exists detailed balancing and therefore

$$V(\mathbf{k}, \mathbf{k}') = W(\mathbf{k}, \mathbf{k}')f_0(1 - f_0') = W(\mathbf{k}', \mathbf{k})f_0'(1 - f_0) = V(\mathbf{k}, \mathbf{k}') \quad (20)$$

in which we have simplified the notation by writing f_0' to denote $f_0(k')$, and which leads to the Fermi function

$$f_0 = [\exp \beta(\epsilon - \zeta) + 1]^{-1}, \qquad \beta = 1/kT. \quad (21)$$

In problems of conductivity we are interested, primarily, in very small displacement from the equilibrium state. Thus we are justified in writing

$$f = f_0 + f_1 = f_0 - Q\partial f_0/\partial \epsilon = f_0 + Qf_0(1 - f_0)\beta \quad (22)$$

which follows from the form of the Fermi function f_0 (22). Hence, if terms involving Q^2 are neglected, which is equivalent to neglecting squares and products of the electric fields and temperature gradients, (19) becomes

$$(\partial f/\partial t)_{\text{coll.}} = \beta \int V(\mathbf{k}, \mathbf{k}')[Q(\mathbf{k}) - Q(\mathbf{k}')]d\mathbf{k}'. \quad (23)$$

b. Operator Form of Transport Equation

The transport equation is then obtained by equating (23) describing the effect of the collisions with the lattice to (14) due to the external field. From (21) one also obtains the relations

$$\nabla_k f_0 = \nabla_k \epsilon \partial f_0/\partial \epsilon = \hbar \mathbf{v} \partial f_0/\partial \epsilon \tag{24}$$

$$\nabla_r f_0 = (\partial f_0/\partial T)\nabla_r T + (\partial f_0/\partial \zeta)\nabla_r \zeta$$

$$\partial f_0/\partial T = -\frac{\epsilon - \zeta}{T}(\partial f_0/\partial \epsilon), \quad \partial f_0/\partial \zeta = -\partial f_0/\partial \epsilon. \tag{25}$$

With the help of these relations, one can write (14) in the steady state up to first orders in f

$$df/dt = -\partial f_0/\partial \epsilon \sum_{i=1}^{3} v_i[eE_i + T\partial/\partial x_i(\zeta/T) + \epsilon/T(\partial T/\partial x_i)]$$

$$+ (\partial f_0/\partial \epsilon)\frac{e}{hc}(\mathbf{v} \times H)\nabla_k Q. \tag{26}$$

The form of this expression suggests the separation of the equation into two parts, one being proportional to ϵ and one independent of it. Thus, if we write for

$$Q = \sum_i \{[eE_i + T\partial/\partial x_i(\zeta/T)]f_{1i} + (1/T)(\partial T/\partial x_i)f_{2i}\}, \tag{27}$$

equations (26) (with $H = 0$) and (23) can be replaced by

$$L(f_{1i}) = v_i \partial f_0/\partial \epsilon = x_{1i} \tag{27a}$$

$$L(f_{2i}) = v_i \partial f_0/\partial \epsilon = x_{2i} \tag{27b}$$

where L is the integral operator defined by (23)

$$L(F) = \beta \int V(\mathbf{k}, \mathbf{k}')[F(\mathbf{k}) - F(\mathbf{k}')]d\mathbf{k}'. \tag{28}$$

The symmetry property (20) leads to the symmetry relation

$$(F; G) = (G; F) \tag{29}$$

where we have defined the inner product

$$(F; G) = \beta \int F(\mathbf{k})L(G)d\mathbf{k}. \tag{30}$$

The truth of this result can be seen by writing

$$(F;G) = \beta \int\int V(\mathbf{k},\mathbf{k}')F(\mathbf{k})[G(\mathbf{k}) - G(\mathbf{k}')]d\mathbf{k}d\mathbf{k}'$$
$$= \beta \int\int V(\mathbf{k},\mathbf{k}')F(\mathbf{k}')[G(\mathbf{k}') - G(\mathbf{k})]d\mathbf{k}d\mathbf{k}'$$

on account of (20). Hence, by addition

$$(F;G) = \beta \int\int V(\mathbf{k},\mathbf{k}')[F(\mathbf{k}) - F(\mathbf{k}')][G(\mathbf{k}) - G(\mathbf{k}')]d\mathbf{k}d\mathbf{k}'$$

which is symmetrical in F and G and hence establishes the relation (29).

c. Effect of the Magnetic Field

In the presence of an external magnetic field Eqs. (27) acquire an extra term arising from the last term in (26) and which can be taken into account by adding to the operator L the operator L'

$$L'(Q) = (e/\hbar c)(\partial f_0/\partial \epsilon)(\mathbf{v} \times \mathbf{H})\nabla_k Q. \tag{31}$$

The operator L' is not symmetric, but satisfies the relation

$$\int FL'(G)d\mathbf{k} = -\int GL'(F)d\mathbf{k} \tag{32}$$

and, on reversal of the magnetic field, H,

$$\int FL_+'(G)d\mathbf{k} = \int GL_-'(F)d\mathbf{k} \tag{33}$$

where L_+' and L_-' denote the operator L' with $+\mathbf{H}$ and $-\mathbf{H}$ respectively in the expression (31). These results follow from the fact that the integral

$$\int \nabla\cdot[(\mathbf{v} \times \mathbf{H})FG\partial f_0/\partial \epsilon]d\mathbf{k}$$

vanishes. It is also to be noted that both $\mathbf{v}$ and $\mathbf{H}$ are solenoidal.

From (33) it is seen that the solutions of the transport equation depend on the direction of the magnetic field. On reversing the magnetic field the electrons will curve in the opposite direction and the condition of microscopic reversibility (20) then becomes

$$V^+(\mathbf{k},\mathbf{k}') = V^-(\mathbf{k}',\mathbf{k}). \tag{34}$$

It is thus possible to divide the set of solutions f_{1i} and f_{2i} into conjugate

sets $f_{j_i}^+$ and $f_{j_i}^-$ $(j = 1, 2)$ for which we have analogous to (27) the following system of equations

$$x_{1_i}^+ = L(f_{1_i}^+) + L_+'(f_{1_i}^+) \tag{35a}$$

$$x_{2_i}^+ = L(f_{2_i}^+) + L_+'(f_{2_i}^+) \tag{35b}$$

together with the corresponding ones for a magnetic field in the opposite direction, which are obtained from (35) by changing all + into − quantities. The system of equations (27) or (35) can be obtained by means of a variational principle, which will be discussed in Section III.

2. The Lattice Distribution Function

The transition probability, $W(\mathbf{k}', \mathbf{k})$, depends mainly on the interaction of the electrons with the lattice. If $n_{\mathbf{q}j}$ denotes the distribution function of the lattice vibration, where $\mathbf{q}$ describes the state of the lattice vibrations, j the polarization ($j = 1, 2$ for transverse and $j = 3$ for longitudinal) the transition probability for a transition $\mathbf{k} \to \mathbf{k}' = \mathbf{k} + \mathbf{q}$ with absorption of a lattice quantum is of the form

$$W(\mathbf{k}, \mathbf{k}') = V(\mathbf{k}, \mathbf{k}')n_{\mathbf{q}}\delta(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}'} + h\nu_{\mathbf{q}}) \tag{36}$$

and for a transition $\mathbf{k} \to \mathbf{k}' = \mathbf{k} - \mathbf{q}$ with emission of a lattice quantum

$$W(\mathbf{k}, \mathbf{k}') = V(\mathbf{k}, \mathbf{k}')(n_{\mathbf{q}} + 1)\delta(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}'} - h\nu_{\mathbf{q}}) \tag{37}$$

where the Dirac δ-functions ensure that energy is conserved in the transition. In other words, (36) vanishes unless $\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}'} + h\nu_{\mathbf{q}} = 0$, and (37) vanishes unless $\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}'} - h\nu_{\mathbf{q}} = 0$. Here $\nu_{\mathbf{q}}$ is the frequency of the lattice wave in the state $\mathbf{q}$.

If the effect of lattice vibrations is neglected, $n_{\mathbf{q}}$ may be replaced by its equilibrium value

$$n_0(\mathbf{q}) = (e^y - 1)^{-1}, \qquad y = h\nu/kT. \tag{38}$$

However, in the general case considered here, both f and $n_{\mathbf{q}j}$ depart from their equilibrium value, and in addition to (12), one has a transport equation for $n_{\mathbf{q}}$

$$(\partial n_{\mathbf{q}}/\partial t)_{\text{coll.}} = \sum_{i=1}^{3} U_i(\partial n_0/\partial T)(\partial T/\partial x_i) \tag{39}$$

where U_i are the components of the lattice wave velocity. Hence, the two functions f and $n_{\mathbf{q}}$ must be determined simultaneously as the solutions of the two Boltzmann equations (12) and (39).

a. Collision Operators

For the two collision operators $(\partial f/\partial t)_{\text{coll.}}$ and $(\partial n/\partial t)_{\text{coll.}}$, respectively, one obtains (taking into account the exclusion principle and the relations (36) and (37) and the reverse transitions $\mathbf{k}' \to \mathbf{k}$):

$$(\partial f/\partial t)_{\text{coll.}} = \int \{[V(\mathbf{k}', \mathbf{k}) f_{\mathbf{k}'}(1-f_{\mathbf{k}})(n_{\mathbf{q}}+1) - V(\mathbf{k}, \mathbf{k}') f_{\mathbf{k}}(1-f_{\mathbf{k}'})n_{\mathbf{q}}] \times \delta(\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{k}'}+h\nu_{\mathbf{q}}) + [V(\mathbf{k}', \mathbf{k}) f_{\mathbf{k}'}(1-f_{\mathbf{k}})n_{\mathbf{q}} - V(\mathbf{k}, \mathbf{k}') f_{\mathbf{k}}(1-f_{\mathbf{k}'})(n_{\mathbf{q}}+1)]\delta(\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{k}'}-h\nu_{\mathbf{q}})\} d\mathbf{k} \quad (40)$$

$$(\partial n/\partial t)_{\text{coll.}} = \int [V(\mathbf{k}', \mathbf{k}) f_{\mathbf{k}'}(1-f_{\mathbf{k}})(n_{\mathbf{q}}+1) - V(\mathbf{k}, \mathbf{k}') f_{\mathbf{k}}(1-f_{\mathbf{k}'})n_{\mathbf{q}}] \times \delta(\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{k}'}+h\nu_{\mathbf{q}}) d\mathbf{k}. \quad (41)$$

In the equilibrium state, these expressions must vanish. Hence, putting $f = f_0$ and $n_{\mathbf{q}} = n_0$, and using the energy equation $\epsilon' = \epsilon \pm h\nu$ we must have

$$V(\mathbf{k}', \mathbf{k}) f_0(\epsilon + h\nu)[1 - f_0(\epsilon)](n_0 + 1) = V(\mathbf{k}, \mathbf{k}') f_0(\epsilon)[1 - f_0(\epsilon + h\nu)] \quad (42)$$

$$V(\mathbf{k}', \mathbf{k}) f_0(\epsilon - h\nu)[1 - f_0(\epsilon)]n_0 = V(\mathbf{k}, \mathbf{k}') f_0(\epsilon)[1 - f_0(\epsilon - h\nu)](n_0 + 1). \quad (43)$$

Both these restrictions are satisfied if $V(\mathbf{k}, \mathbf{k}')$ satisfies the symmetry relation (20) which expresses the principle of microscopic reversibility for the present problem and forms the basis of the Kelvin relations to be discussed later.

If we now expand $n_{\mathbf{q}j}$ in the same manner as has been done for f

$$n_{\mathbf{q}j} = n_0 - N_j \beta \partial n_0/\partial y = n_0 + \beta n_0(n_0 + 1)N_j \quad (44)$$

the exact expressions (33) and (34) may be simplified. Keeping only terms linear in Q and N_j we obtain

$$(\partial f/\partial t)_{\text{coll.}} = \beta \int V(\mathbf{k}, \mathbf{k}')[n_0(\mathbf{q}) f_0(\mathbf{k})(1-f_0(\mathbf{k}'))]\delta(\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{k}'}+h\nu_{\mathbf{q}}) \times [Q(\mathbf{k}') - Q(\mathbf{k}) - N(\mathbf{q})] + n_0(\mathbf{q}) f_0(\mathbf{k}')(1-f_0(\mathbf{k})) \times \delta(\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{k}'}-h\nu_{\mathbf{q}})[Q(\mathbf{k}') - Q(\mathbf{k}) + N(\mathbf{q})] d\mathbf{k} \quad (45)$$

$$(\partial n/\partial t)_{\text{coll.}} = \beta \int V(\mathbf{k}, \mathbf{k}') n_0(\mathbf{q})[1-f_0(\mathbf{k}')]\delta(\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{k}'}+h\nu_{\mathbf{q}}) \times [Q(\mathbf{k}') - Q(\mathbf{k}) - N(\mathbf{q})] d\mathbf{k} \quad (46)$$

where the left-hand side of these expressions is given by (84) and (88), respectively. They are now a system of integro-differential equations for the two quantities Q and N_j.

b. Operator Form

The system of equations, (45) and (46) can be written in operator form

$$x_{1i} = v_i \partial f_0/\partial \epsilon = L_1(f_{1i}) + L_2(g_{1i}) \tag{47a}$$

$$y_{1i} = 0 = L_3(f_{1i}) + L_4(g_{1i}) \tag{47b}$$

$$x_{2i} = \epsilon v_i \partial f_0/\partial \epsilon = L_1(f_{2i}) + L_2(g_{2i}) \tag{47c}$$

$$y_{2i} = -U_i T \partial n_0/\partial T = L_3(f_{2i}) + L_4(g_{2i}) \tag{47d}$$

where the function N has been separated into two parts similar to Q (*cf.* 27)

$$N = \sum_i \{[eE_i + T\partial/\partial x^i(\zeta/T)]g_{1i} + (1/T\,\partial T/\partial x^i)g_{2i}\}. \tag{48}$$

The integral operators L_i (L = 1 to 4) are given by

$$L_1(F) = \beta \int V(\mathbf{k}, \mathbf{k}')[S(\mathbf{k}, \mathbf{k}') + S(\mathbf{k}', \mathbf{k})][F(\mathbf{k}) - F(\mathbf{k}')]d\mathbf{k}' \tag{49a}$$

$$L_2(F) = \beta \int V(\mathbf{k}, \mathbf{k}')[S(\mathbf{k}, \mathbf{k}')F(\mathbf{q}) - S(\mathbf{k}', \mathbf{k})F(\mathbf{q})]d\mathbf{k}' \tag{49b}$$

$$L_3(F) = \beta \int V(\mathbf{k}, \mathbf{k}')S(\mathbf{k}, \mathbf{k}')[F(\mathbf{k}) - F(\mathbf{k}')]d\mathbf{k} \tag{49c}$$

$$L_4(F) = \beta \int V(\mathbf{k}, \mathbf{k}')S(\mathbf{k}, \mathbf{k}')F(\mathbf{q})d\mathbf{k} \tag{49d}$$

(It is important to note that $\mathbf{q} = \mathbf{k}' - \mathbf{k}$ is to be kept constant in (49*c*) and (49*d*) which are therefore functions of $\mathbf{q}$.) $S(\mathbf{k}, \mathbf{k}')$ is given by

$$S(\mathbf{k}, \mathbf{k}') = n_0(\mathbf{q})f_0(\mathbf{k})[1 - f_0(\mathbf{k}')]\delta(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}'} + h\nu_{\mathbf{q}}) \tag{50}$$

corresponding to an absorption of a lattice quantum $\mathbf{k} \rightarrow \mathbf{k}' = \mathbf{k} + \mathbf{q}$, (36), while $S(\mathbf{k}', \mathbf{k})$ corresponds to the emission of a lattice quantum $\mathbf{k} \rightarrow \mathbf{k}' = \mathbf{k} - \mathbf{q}$ and has the transition probability given by (37).

For compactness of notation, let us now introduce the generalized inner product analogue to (30)

$$(F; G)_i = \int F(\mathbf{k})L_i(G)d\mathbf{k} \qquad i = 1, 2, 3, 4 \tag{51}$$

where the subscript i indicates the operator L_i defined by (49). The symmetry property (20) leads to the symmetry relations

$$(F; G)_1 = (G; F)_1 \tag{52a}$$

$$(F; G)_2 = (G; F)_3 \tag{52b}$$

$$(F; G)_4 = (G; F)_4. \tag{52c}$$

The property (52*a*) follows from (29), while (52*c*) follows at once from the form of $L_4(F)$ (49*d*). To establish (52*b*) we write

$$(F;G)_2 = \int F(k)L_2(G)d\mathbf{k}$$
$$= \beta \int\int V(\mathbf{k},\mathbf{k}')F(\mathbf{k})[S(\mathbf{k},\mathbf{k}')G(\mathbf{q}) - S(\mathbf{k},\mathbf{k}')G(\mathbf{q})]d\mathbf{k}d\mathbf{k}'$$
$$= \beta \int\int V(\mathbf{k},\mathbf{k}')[S(\mathbf{k},\mathbf{k}')F(\mathbf{k})G(\mathbf{q}) - S(\mathbf{k},\mathbf{k}')F(\mathbf{k}')G(\mathbf{q})]d\mathbf{k}d\mathbf{k}'$$

since $V(\mathbf{k},\mathbf{k}') = V(\mathbf{k}',\mathbf{k})$. Hence

$$(F;G)_2 = \beta \int\int V(\mathbf{k},\mathbf{k}')S(\mathbf{k},\mathbf{k}')[F(\mathbf{k}) - F(\mathbf{k}')]G(\mathbf{q})d\mathbf{k}d\mathbf{k}'$$
$$= \beta \int G(\mathbf{q})L(F)d\mathbf{k}' = (G;F)_2$$

which completes the proof.

c. Effect of a Magnetic Field

As in the case without lattice vibrations taken into account, the presence of an external magnetic field introduces an extra term in the Boltzmann equations (47) which again can be represented by an operator L' defined by (31) and possessing the properties (32) and (33). The modified equations are then given by

$$x_{1i}^{+} = L_1(f_{1i}^{+}) + L_2(g_{1i}) + L_{+}'(f_{1i}^{+}) \tag{53a}$$
$$y_{1i} = L_3(f_{1i}^{+}) + L_4(g_{1i}) \tag{53b}$$
$$x_{2i}^{+} = L_1(f_{2i}^{+}) + L_2(g_{2i}) + L_{+}'(f_{2i}^{+}) \tag{53c}$$
$$y_{2i} = L_3(f_{2i}^{+}) + L_4(g_{2i}) \tag{53d}$$

with corresponding ones for a magnetic field in the opposite direction, which are again obtained from (53) by changing all + to − quantities. As L' does not operate on N, it is not necessary to separate g_{1i} and g_{2i} into two conjugate sets, as must be done for f_{1i} and f_{2i}. The solution of this more complicated system of equations can also be obtained by a variational principle to be discussed in Section III.

3. Interband Scattering

In some cases of physical interest, notably in semiconductors, one has more than one energy surface, and it becomes necessary to consider the effect of transitions between bands. Before generalizing (27) to a

multiple band model, we note that the operator equations (27) can be written as

$$X_{\mathbf{k}} = \sum_{\mathbf{k}'} L_{\mathbf{k}\mathbf{k}'} J_{\mathbf{k}'} \tag{54}$$

showing the connection between the forces $X_{\mathbf{k}}$ and currents $J_{\mathbf{k}}$. From the properties of $L(Q)$ we can also infer that the coefficients $L_{\mathbf{k}\mathbf{k}'}$ form a positive definite symmetric matrix, with the symmetry relations $L_{\mathbf{k}\mathbf{k}'} = L_{\mathbf{k}'\mathbf{k}}$. Indeed, this last result forms the basis of the Kelvin relations amongst the thermoelectric coefficients to be discussed later. The summation in (54) is to be carried out over all the states $\mathbf{k}$. In order to take into account the other bands we simply extend the summation in (54) over the appropriate states. Thus the system of equations (27) has to be replaced by

$$\sum_n L_{mn}(f_{1i}^n) = v_i \partial f_0/\partial \epsilon_m = x_{1i}^m \tag{55a}$$

$$\sum_n L_{mn}(f_{2i}^n) = v_i \epsilon_m \partial f_0/\partial \epsilon_m = x_{2i}^m \tag{55b}$$

where the operator L_{mn} is defined by

$$L_{mn}(F) = \beta \int V(\mathbf{k}_m, \mathbf{k}_n)[F(\mathbf{k}_m) - F(\mathbf{k}_n)]d\mathbf{k}_n \tag{56}$$

and m and n denote the possible energy bands. Thus the effect of a scattering from band m to n is taken into account by the operator L_{mn} acting on the various states $F^n = F(\mathbf{k}_n)$ and producing a coupling between them.

The symmetry property (20) in our present notation becomes

$$V(\mathbf{k}_m, \mathbf{k}_n) = V(\mathbf{k}_n, \mathbf{k}_m) \tag{57}$$

and again leads to the symmetry relation

$$(F; G) = (G; F), \tag{29'}$$

where, however, now the inner product $(F; G)$ is a sum of terms of the type $(F^m; G^n)$ such that

$$(F; G) = \sum_{m,n} (F^m; G^n) \tag{58}$$

where we defined the product $(F^m; G^n)$ by

$$(F^m; G^n) = \int F^m L_{mn}(G^n) d\mathbf{k}_m. \tag{59a}$$

The proof of the generalized symmetry relation is obtained by writing again

$$(F;G) = \beta \sum_{m,n} \int\int V(\mathbf{k}_m, \mathbf{k}_n) F(\mathbf{k}_m)[G(\mathbf{k}_m) - G(\mathbf{k}_n)] d\mathbf{k}_m d\mathbf{k}_n \quad (59b)$$

$$= \beta \sum_{m,n} \int\int V(\mathbf{k}_m, \mathbf{k}_n) F(\mathbf{k}_n)[G(\mathbf{k}_m) - G(\mathbf{k}_n)] d\mathbf{k}_m d\mathbf{k}_n \quad (59c)$$

on account of (57). Hence by addition

$$(F;G) = \beta \sum_{m,n} \int\int V(\mathbf{k}_m, \mathbf{k}_n)[F(\mathbf{k}_m) - F(\mathbf{k}_n)] \times [G(\mathbf{k}_m) - G(\mathbf{k}_n)] d\mathbf{k}_m d\mathbf{k}_n \quad (59d)$$

which is symmetrical in F and G, as for each term $(F^m; G^n)$ in the sum we have a corresponding one $(F^n; G^m)$, and thus establishes the truth of (29′).

a. Effect of the Magnetic Field

In the presence of an external magnetic field, the equations (55) acquire an extra term, or rather a series of terms, of the type (31). Thus the new modified operator equations are now given by

$$x_{1i}{}^{m+} = \sum_n L_{mn}(f_{1i}{}^{n+}) + L_m'(f_{1i}{}^m) \quad (60a)$$

$$x_{2i}{}^{m+} = \sum_n L_{mn}(f_{2i}{}^{n+}) + L_m'(f_{2i}{}^{m+}) \quad (60b)$$

where the $+$ and $-$ signs again denote the direction of the magnetic field, and where the sets of functions $f_{1i}{}^n$ and $f_{2i}{}^n$ are now divided into conjugate sets $f_{1i}{}^{\pm}$ and $f_{2i}{}^{n\pm}$ as before, and the operator L_m' defined by

$$L_m'(F^m) = e/\hbar c \; \partial f_0/\partial \epsilon_m (\mathbf{v}_m \times \mathbf{H}) \nabla_{k_m} F^m.$$

It should be noted that the operator L_m' describes the effect of the magnetic field on the band in question, and therefore only has a free index m, characteristic of the particular band.

b. Presence of Lattice Distribution Function

We have seen that the presence of a non-equilibrium lattice distribution function modifies the transport equation in two ways. First, the lattice distribution function satisfies a Boltzmann equation of the type (39) and secondly, the original Boltzmann equation for the carried distribution function acquires additional terms. The transition from band m to n is characterized by the emission (or absorption) of a lattice

quantum according to $\mathbf{k}_n = \mathbf{k}_m \pm \mathbf{q}$, and hence the interband scattering can be taken into account formally by replacing $\mathbf{k}$ and $\mathbf{k}'$ in the expressions for the integral operators L_i (49) by $\mathbf{k}_m$ and $\mathbf{k}_n$, respectively. We note also that in the case of L_3 and L_4 summation must be carried out over m and n, as the lattice function depends on $\mathbf{q}$, and is therefore independent of m and n.

The system of equations (47) is then replaced analogous to (55) by the following system of operator equations

$$x_{1i}{}^m = \sum_n L_{mn}{}'(f_{1i}{}^n) + \sum_n L_{mn}{}^2(g_{1i}) \tag{61a}$$

$$0 = \sum_{mn} L_{mn}{}^3(f_{1i}{}^n) + \sum_{mn} L_{mn}{}^4(g_{1i}) \tag{61b}$$

$$x_{2i}{}^m = \sum_n L_{mn}{}'(f_{2i}{}^n) + \sum_{mn} L_{mn}{}^2(g_{2i}) \tag{61c}$$

$$y_{2i}{}^m = \sum_{mn} L_{mn}{}^3(f_{2i}{}^n) + \sum_{mn} L_{mn}{}^4(g_{2i}) \tag{61d}$$

where the integral operators $L_{mn}{}^i$ ($i = 1$ to 4) are now given by

$$L_{mn}{}^1(F^n) = \beta \int V(\mathbf{k}_m, \mathbf{k}_n)[S(\mathbf{k}_m, \mathbf{k}_n) + S(\mathbf{k}_n, \mathbf{k}_m)] \times [F(\mathbf{k}_m) - F(\mathbf{k}_n)]d\mathbf{k}_n \tag{62a}$$

$$L_{mn}{}^2(G) = \beta \int V(\mathbf{k}_m, \mathbf{k}_n)[S(\mathbf{k}_m, \mathbf{k}_n)G(\mathbf{q}) - S(\mathbf{k}_n, \mathbf{k}_m)G(\mathbf{q})]d\mathbf{k}_n \tag{62b}$$

$$L_{mn}{}^3(F^n) = \beta \int V(\mathbf{k}_m, \mathbf{k}_n)S(\mathbf{k}_m, \mathbf{k}_n)[F(\mathbf{k}_m) - F(\mathbf{k}_n)]d\mathbf{k}_m d\mathbf{k}_n \tag{62c}$$

$$L_{mn}{}^4(G) = \beta \int V(\mathbf{k}_m, \mathbf{k}_n)S(\mathbf{k}_m, \mathbf{k}_n)G(\mathbf{q})d\mathbf{k}_m d\mathbf{k}_n \tag{62d}$$

with

$$S(\mathbf{k}_m, \mathbf{k}_n) = n_0(q)f_0(\mathbf{k}_m)[1 - f_0(\mathbf{k}_n)]\delta(\epsilon_m - \epsilon_n + h\nu_q) \tag{63}$$

for an absorption of a lattice quantum $\mathbf{k}_m \to \mathbf{k}_n = \mathbf{k}_m + \mathbf{q}$, and $S(\mathbf{k}_n, \mathbf{k}_m)$ corresponding to the emission of a lattice quantum according to $\mathbf{k}_m \to \mathbf{k}_n = \mathbf{k}_m - \mathbf{q}$. We note that the operators $L_{mn}{}^3$ and $L_{mn}{}^4$ have to be summed over both indices m and n, as the lattice distribution functions g_{1i} and g_{2i} are independent of the interband interaction. This can also be seen by inspection of their operator forms, (59c) and (59d), which are integrated over both $d\mathbf{k}_m$ and $d\mathbf{k}_n$.

The appropriate generalized inner products analogous to (51) are now defined as

$$(F;G)_i = \sum_{mn} (F^m;G^n)_{mn}{}^i \qquad i = 1, 3 \tag{64a}$$

$$(F;G)_i{}' = \sum_{mn} (F^m;G)_{mn}{}^{i'} \qquad i = 2, 4 \tag{64b}$$

with

$$(F^m; G^n)_{mn}{}^i = \int F(\mathbf{k}_m) L_{mn}{}^i(G^n) d\mathbf{k}_m \qquad i = 1, 3 \qquad (65a)$$

$$(F^m; G)_{mn}{}^{i'} = \int F(\mathbf{k}_m) L_{mn}{}^i(G) d\mathbf{k}_m \qquad i = 2, 4. \qquad (65b)$$

The need for differentiating between the two cases a and b arises from the fact that in the second case the functions G represent the lattice distribution function and are independent of the band, as can be seen by inspection of the corresponding operators L^2 and L^4 (62b) and (62c). The superscript i gain indicates the operator L^i, while the subscript mn shows the bands.

The symmetry property (57) again leads to symmetry relations of the type (52).

$$(F; G)_1 = (G; F)_1 \qquad (52a')$$

$$(F; G)_2 = (G; F)_3 \qquad (52b')$$

$$(F; G)_4 = (G; F)_4 \qquad (52c')$$

the first of which follows from (29′) while the last can be reduced from the form of L^4. To establish (52b') we write again

$$\begin{aligned}(F; G)_2 &= \sum_{mn} \int F(\mathbf{k}_m) L_{mn}{}^2(G) d\mathbf{k}_m \\ &= \beta \sum_{mn} \int\int V(\mathbf{k}_m, \mathbf{k}_n) F(\mathbf{k}_m) \\ &\qquad \times [S(\mathbf{k}_m, \mathbf{k}_n) G(\mathbf{q}) - S(\mathbf{k}_n, \mathbf{k}_m) G(\mathbf{q})] d\mathbf{k}_m d\mathbf{k}_n \\ &= \beta \sum_{mn} \int\int V(\mathbf{k}_m, \mathbf{k}_n) \\ &\qquad \times [S(\mathbf{k}_m, \mathbf{k}_n) F(\mathbf{k}_m) G(\mathbf{q}) - S(\mathbf{k}_m, \mathbf{k}_n) F(\mathbf{k}_n) G(\mathbf{q})] d\mathbf{k}_m d\mathbf{k}_n\end{aligned}$$

since $V(\mathbf{k}_m, \mathbf{k}_n) = V(\mathbf{k}_n, \mathbf{k}_m)$. Hence

$$\begin{aligned}(F; G)_2 &= \beta \sum_{mn} \int\int V(\mathbf{k}_m, \mathbf{k}_n) S(\mathbf{k}_m, \mathbf{k}_n) G(\mathbf{q}) [F(\mathbf{k}_m) - F(\mathbf{k}_n)] d\mathbf{k}_m d\mathbf{k}_n \\ &= \beta \sum_{mn} \int\int G(\mathbf{q}) L_{mn}{}^3(F) d\mathbf{k} = (G; F)_3\end{aligned}$$

which completes the proof. In the case of $i = 2$ and 4, the functions G do not have a superscript, as L_{mn} operates on the lattice distribution function in these instances which follows from the definition of the appropriate operators (62b) and (62d).

Finally, if in addition to non-equilibrium lattice vibrations we also have an external magnetic field as described in (a) of this section, the

Boltzmann equations (61) will acquire extra terms of the type (57) and result in conjugate sets of solutions $f_{1i}^{n\pm}$ and $f_{2i}^{n\pm}$, do not affect the functions g_{1i} and g_{2i}, as mentioned before. The modified system of equations is then given by

$$x_{1i}^{m+} = \sum_n L_{mn}^{1}(f_{1i}^{n+}) + \sum_n L_{mn}^{2}(g_{1i}) + L_m'(f_{1i}^{m+}) \tag{66a}$$

$$0 = \sum_{m,n} L_{mn}^{3}(f_{1i}^{n+}) + \sum_{mn} L_{mn}^{4}(g_{1i}) \tag{66b}$$

$$x_{2i}^{m+} = \sum_n L_{mn}^{1}(f_{2i}^{n+}) + \sum_n L_{mn}^{2}(g_{2i}) + L_m'(f_{2i}^{m+}) \tag{66c}$$

$$y_{2i} = \sum_{m,n} L_{mn}^{3}(f_{1i}^{n+}) + \sum_{mn} L_{mn}^{4}(g_{2i}) \tag{66d}$$

with corresponding ones for a magnetic field in the opposite direction, which are again obtained from (66) by changing all + to − quantities. The solution of the various systems of equations with increasing complexity, (55, 60, 61 and 66) can also be obtained by a variational principle which will be described in Section III.

III. VARIATIONAL PRINCIPLE

1. Variational Principle for Distribution Function

It has been shown by Kohler that the Boltzmann equation for the particle distribution function (*cf.* 27) is equivalent to a variational principle, which preserves the inherent symmetry properties. An application of the Ritz method enables one then to obtain a series solution for the distribution function and the transport coefficients in terms of infinite determinants, which can be evaluated to any required degree of accuracy.

Consider an equation of the form

$$L(F) = X \tag{67}$$

where L is an integral operator of the type (28) with positive definite kernel $V(\mathbf{k}, \mathbf{k}')$, and X a known function, such as given by (27). It can be shown that (67) can be replaced by a variational principle. For compactness of notation, let us introduce in addition to the inner product $(F; G)$ defined previously (30)

$$(F; G) = \int FL(G)d\mathbf{k}, \tag{68}$$

the product

$$(F, X) = \int FXd\mathbf{k}. \tag{69}$$

Then, if F is a solution of (67) and G any function which satisfies the relation

$$(G;G) = (G, X) \tag{70}$$

then of all functions G satisfying (70), F is the one which makes $(G;G)$ a maximum. The proof of this statement depends on the relation $(F;G) = (G;F)$ and $(G;G) \geq 0$ which are consequences of the symmetry property (20) and the fact that $V(\mathbf{k}, \mathbf{k}')$ is essentially positive definite. These relations have been derived in the last section, (29 and 29′). To prove the variational principle we now proceed as follows. If we multiply (67) by G and integrate over $\mathbf{k}$-space, we find

$$(G;F) = (G, X) = (G;G) \tag{71}$$

on account of (70). From (29′) we have $(F - G; F - G) \geq 0$. But

$$0 \leq (F - G; F - G) = (F;F) - (F;G) - (G;F) + (G;G)$$

$$(F - G; F - G) = (F;F) - 2(G;F) + (G;G) \quad \text{on account of (29)}$$

$$= (F;F) - (G;G) \quad \text{on account of (29′)}$$

so that finally

$$(F;F) \geq (G;G) \tag{72}$$

which proves our assertion.

Thus our variational function is the inner product $(F;F)$ subject to the subsidiary condition

$$(F;F) = (F, X). \tag{73}$$

Consider now the new variational function

$$H = (F;F) + \lambda(F, X) \tag{74}$$

where λ is a Lagrangian multiplier. Variation of H with respect to δF results in

$$\delta H = \int dF[2L(F) + \lambda X]d\mathbf{k} = 0$$

which is equivalent to (67) provided we set $\lambda = -2$.

Applying these results to the system of equations (27) we introduce the variational functions H_l $(l = 1, 2)$ (74)

$$H_1 = (f_{1i}; f_{1j}) + \lambda_1(f_{1i}, x_{1j}) \tag{74a}$$

$$H_2 = (f_{2i}; f_{2j}) + \lambda_2(f_{2i}, x_{2j}) \tag{74b}$$

which on variation with respect to f_{1i} and f_{2i} respectively, lead to the system of equations (27) provided we set $\lambda_1 = \lambda_2 = -2$.

a. Solution of the Variational Principle

The variational principle (74) is then solved by the Ritz method. In order to do this we expand f_{1i} and f_{2i} in a power series in ϵ (or $\mathfrak{x} = \epsilon - \zeta$ which might be more convenient)

$$f_{1i} = \sum_s a_s \mathfrak{x}^s k_i \tag{75a}$$

$$f_{2i} = \sum_s b_s \mathfrak{x}^s k_i \tag{75b}$$

where the coefficients a_s and b_s are to be determined by the Ritz method. From (68) we have

$$(f_{1i}; f_{1j}) = \sum_{r,s} a_r a_s d_{rs}^{ij} \tag{76a}$$

$$(f_{2i}; f_{2j}) = \sum_{r,s} b_r b_s d_{rs}^{ij} \tag{76b}$$

where

$$d_{rs}^{ij} = (\mathfrak{x}^r k_i; \mathfrak{x}^s k_j) = d_{sr}^{ij} \tag{77}$$

while the subsidiary conditions (73) become

$$\sum_{r,s} d_{rs}^{ij} a_r a_s = \sum_r \alpha_r^{ij} a_r \tag{78a}$$

$$\sum_{r,s} d_{rs}^{ij} b_r b_s = \sum_r \beta_r^{ij} a_r \tag{78b}$$

where we have defined

$$\alpha_r^{ij} = (\mathfrak{x}^r k_i, x_{1j}), \qquad \beta_r^{ij} = (\mathfrak{x}^r k_i, x_{2j}). \tag{79}$$

The variational functions H (74*a* and 74*b*) then become

$$H_1^{ij} = \sum_{r,s} a_r a_s d_{rs}^{ij} + \lambda_1 \sum_r \alpha_r^{ij} a_r \tag{80a}$$

$$H_2^{ij} = \sum_{r,s} b_r b_s d_{rs}^{ij} + \lambda_2 \sum_r \beta_r^{ij} b_r. \tag{80b}$$

Variation with respect to a_s and b_s then results in

$$2 \sum_s a_s d_{rs}^{ij} + \lambda_1 \alpha_r^{ij} = 0 \tag{81a}$$

$$2 \sum_s b_s d_{rs}^{ij} + \lambda_2 \beta_r^{ij} = 0. \tag{81b}$$

The Lagrangian multipliers λ_1 and λ_2 are found to be $\lambda_1 = \lambda_2 = -2$ by multiplying (81) by a_r and b_r, respectively and comparing with the subsidiary conditions (78). This, then, results in the following system of equations

$$\sum_s a_s d_{rs}^{ij} = \alpha_r^{ij} \tag{82a}$$

$$\sum_s b_s d_{rs}^{ij} = \beta_r^{ij} \tag{82b}$$

for the unknowns a_r and b_r. The distribution function f_{1i} and f_{2i} can then be obtained in terms of infinite determinants. Let us denote by D the determinant whose elements are d_{rs}, thus $D = |d_{rs}|$ and D_α^i and D_β^i the determinants obtained from D by adding a column of α_r and β_r, respectively, and a row consisting of $\mathfrak{X}^r k_i$ the last diagonal element being zero. Then, it follows from (82) and (75) and the property of determinants that f_{1i} and f_{2i} are given by

$$f_{1i} = \sum a_s \mathfrak{X}^s k_i = D_\alpha / D \tag{84a}$$

$$f_{2i} = \sum a_s \mathfrak{X}^s k_i = D_\beta / D. \tag{84b}$$

b. Entropy Production

There exists a simple relationship between the variational function $(F; F)$ and the rate of entropy production. The transport equation for the carrier distribution function can be written in the general form (*cf.* 13)

$$\partial f/\partial t]_{\text{coll.}} = \partial f/\partial t + \mathbf{v} \cdot \nabla_r f + e/\hbar \mathbf{E} \cdot \nabla_k f. \tag{85}$$

Multiplying this equation by $\ln f$ and adding to it the corresponding equation, where f has been replaced by $(1 - f)$, (85) can be written after integrating over $d\mathbf{k}$

$$\partial S/\partial t + \text{div. } \mathbf{s} = -k \int [\ln f - \ln (1 - f)] \partial f/\partial t]_{\text{coll.}} d\mathbf{k}, \tag{86}$$

where S and $\mathbf{s}$ are the entropy density and entropy current density, respectively, and defined by

$$S = -k \int [f \ln f + (1 - f) \ln (1 - f)] d\mathbf{k} \tag{87a}$$

$$\mathbf{s} = -k \int [f \ln f + (1 - f) \ln (1 - f)] \mathbf{v} d\mathbf{k}. \tag{87b}$$

In obtaining (86) we have made use of the vector indentities

$$\text{div } (A\mathbf{v}) = \mathbf{v} \cdot \nabla A + A \text{ div } \mathbf{v}. \tag{88}$$

If we now use the expansion for f given by (22), then the right-hand side of (86) can be written up to first orders in Q as

$$-1/T \int [Q(\mathbf{k}) - \epsilon/kT] L(Q) d\mathbf{k}. \tag{89}$$

As the energy remains constant during the type of collisions considered here, the term proportional to the energy can be omitted, so that we have finally

$$\partial S/\partial t + \operatorname{div} \mathbf{s} = -1/T \int Q(\mathbf{k}) L(Q) d\mathbf{k} = -1/T(Q;Q) \tag{90}$$

which, apart from a factor $(-1/T)$, is just the variational function $(Q;Q)$ for the problem. Moreover, as we can prove from the H-theorem that is always smaller than zero, it follows that $(Q;Q)$ is positive definite, as shown earlier. For the steady state, of course, $\partial S/\partial t + \operatorname{div} \mathbf{s} = 0$ and hence the over-all change of f is also zero.

c. Effect of Magnetic Field

In the presence of an external magnetic field, it has been shown that the Boltzmann transport equation acquires an extra term which we represented by the operator L' defined by (87), so that we can write the modified equation in operator form as

$$X^+ = L_1(F^+) + L_+'(F^+). \tag{91}$$

The entropy production due to magnetic scattering according to (90) is then given by

$$\partial S/\partial t]_{\text{mag.}} = -1/T \int FL'(F) d\mathbf{k} = 0 \tag{92}$$

which is seen to vanish on account of (32), so that there is no explicit contribution from magnetic scattering to the production of entropy. The only effect of the magnetic field is to bend the electrons without changing the order of the distribution.

On account of the symmetry relation (33) on reversal of the magnetic field $\mathbf{H}$, it is possible to obtain a variational principle for this case also, which will give rise to the correct transport equation (35), although the variational function does not contribute to the entropy production as shown above. Analogous to (74) let us define a variational function H by

$$H = (F^-; F^+) + (F^-; F^+)' + \lambda(F^-, X) \tag{93}$$

where $(F^-; F^+)$ is the inner product defined by (68) and the + and − signs indicate the appropriate solution $F^\pm$ depending on the direction of the field, and $(F^-; G^+)'$ denotes the inner product

$$(F^-; G^+)' = \int FL_+'(G)d\mathbf{k} = (G^+; F^-)' \tag{94}$$

where the equality follows from (34). Hence, on variation this results in

$$\delta H = (\delta F^-; F^+) + (F^-; \delta F^+) + (\delta F^-; F^+)' + (F^-; \delta F^+)' + \lambda(\delta F^-, X) = 0. \tag{95}$$

Now, since L is independent of H, the first two terms in (95) are equal, while the next two are equal on account of (94), so that (95) will reduce to (94) provided we choose again $\lambda = -2$.

Applying this result to the system of equations (35) we now have for our new variational functions H_k $(k = 1, 2)$

$$H_{1i} = (f_{1i}^-; f_{1i}^+) + (f_{1i}^-; f_{1i}^+)' - 2(f_{1i}, x_{1i}^+) \tag{96a}$$

$$H_{2i} = (f_{2i}^-; f_{2i}^+) + (f_{2i}^-; f_{2j}^+)' - 2(f_{2i}^-; x_{2j}^+). \tag{96b}$$

The solution is again obtained by the Ritz method, where now the functions $f_{1i}^\pm$ and $f_{2i}^\pm$ are expanded according to

$$f_{1i}^\pm = \sum_s a_s^\pm \mathfrak{X}^s k_i, \qquad f_{2i}^\pm = \sum_s b_s^\pm \mathfrak{X}^s k_i \tag{97}$$

and the new coefficients d_{rs} are defined by

$$d_{rs}^{ij} = (\mathfrak{X}^r k_i; \mathfrak{X}^s k_j) + (\mathfrak{X}^r k_i; \mathfrak{X}^s k_j)'. \tag{98}$$

Everything goes through as before, and we finally obtain two systems of equations of the type (82*a* and 82*b*) depending whether the + or − sign has been used for the unknowns a and b (and coefficients d, α and β).

2. Generalized Variational Principle

We have seen that the Boltzmann equation for the particle distribution function is equivalent to a variational principle. It can be shown that in the presence of a lattice distribution function the resulting equations (47) can be complicated in this case due to the presence of additional integral operators and the lattice distribution function.

Let Q and N be two functions satisfying the integral equations

$$\begin{aligned} X &= L_1(Q) + L_2(N) \\ Y &= L_3(Q) + L_4(N) \end{aligned} \tag{99}$$

where L_i ($i = 1$ to 4) are the integral operators defined by (49). Furthermore, let R and M be two functions satisfying the relations

$$\begin{aligned}(R, X) &= (R; R)_1 + (R; M)_2 \\ (M, Y) &= (M; R)_3 + (M; M)_4\end{aligned} \tag{100}$$

where the inner product $(F; G)_i$ has been defined by (51) and (F, X) by (69). As Q and N satisfy the equations (99) we have on multiplying by R and M, respectively, and integrating over $\mathbf{k}$

$$\begin{aligned}(R, X) &= (R; Q)_1 + (R; N)_2 \\ (M, Y) &= (M; Q)_3 + (M; N)_4\end{aligned} \tag{101}$$

from which it follows that the right-hand sides of the system of equations (100) and (101) are equal.

Consider now the linear combination of expressions of the type $(F; G)_i$ which can be formed for these functions

$$\langle F_1F_2; G_1G_2\rangle = (F_1; F_2)_1 + (F_1; G_2)_2 + (G_1; F_2)_3 + (G_1; G_2)_4 \tag{102}$$

where F_j and G_j ($j = 1, 2$) are any functions. From the symmetry relations (52) it follows that it is symmetric on interchange of the index 1 with 2, that is:

$$\langle F_2F_1; G_2G_1\rangle = \langle F_1F_2; G_1G_2\rangle. \tag{103}$$

Moreover, on setting $F_1 = F_2 = F$ and $G_1 = G_2 = G$, the resulting expression is positive definite, that is,

$$\langle F; G\rangle = (F; F)_1 + (F; G)_2 + (G; F)_3 + (G; G)_4 \geq 0. \tag{104}$$

The truth of this statement can be established by the use of the symmetry relations and rearranging some of the terms. For example, due to the symmetry in $\mathbf{k}$ and $\mathbf{k}'$ the first term in (104) can be written following the argument leading to (29′) as

$$(F; F)_1 = \beta \int\int V(\mathbf{k}, \mathbf{k}')S(\mathbf{k}, \mathbf{k}')[F(\mathbf{k}) - F(\mathbf{k}')]^2 d\mathbf{k}d\mathbf{k}'.$$

As G is an odd function of its argument, the second term in (104) becomes, on making use of the form of the operator L_2 (49n),

$$(F; G)_2 = \beta \int\int V(\mathbf{k}, \mathbf{k}')S(\mathbf{k}, \mathbf{k}')[F(\mathbf{k}) - F(\mathbf{k}')]G(\mathbf{q})d\mathbf{k}d\mathbf{k}'$$

and so finally, the whole expression (104) is given by

$$\langle F; G\rangle = \beta \int\int [F(\mathbf{k}) + G(\mathbf{q})]^2 P_a(\mathbf{k}, \mathbf{k}')d\mathbf{k}d\mathbf{k}'. \tag{105}$$

This is clearly a positive definite function, as the probability $P_s(\mathbf{k}, \mathbf{k}') = V(\mathbf{k}, \mathbf{k}')S(\mathbf{k}, \mathbf{k}')$ is of necessity a positive definite function and the integrand in (105) consists of a square multiplied by it. This, then completes the proof of our assertion. It might be pointed out that (105) is simply related to the total entropy production for this process. If we now replace the function F by $Q - R$ and G by $N - M$ in (104) it can be shown that the following inequality holds

$$(Q;Q)_1 + (Q;N)_2 + (N;Q)_3 + (N;N)_4 \geq (R;R)_1 + (R;M)_2 + (M;R)_3 + (M;M)_4 \quad (106)$$

where we have made use of the relations (101) and (102) and the symmetry properties (52). This enables us now to state the following theorem:

> If Q and N are two functions satisfying the integral equations (99), and if R and M are two arbitrary functions satisfying the relations (100), then of all these functions Q and N are the ones making (104) a maximum subject to the subsidiary conditions (101). Thus our variational function is the generalized inner product $\langle Q; N\rangle$ subject to the subsidiary conditions
>
> $$\begin{aligned}(Q, X) &= (Q;Q)_1 + (Q;N)_2 \\ (N, Y) &= (N;Q)_3 + (N;N)_4.\end{aligned} \quad (107)$$

This theorem can also be applied to the system of equations (47) simply by replacing Q by f_{1i} $(l = 1, 2)$. The expression to be maximized is then given by

$$\langle f_{1i}; g_{1j}\rangle = (f_{1i};f_{1j})_1 + (f_{1i};g_{1j})_2 + (g_{1i};f_{1j})_3 + (g_{1i};g_{1j})_4 \qquad l = 1, 2 \quad i, j = 1, 2, 3 \quad (108)$$

and the subsidiary conditions (107) replaced by

$$\begin{aligned}(f_{1i}, x_{1j}) &= (f_{1i};f_{1j})_1 + (f_{1i};g_{1j})_2 \\ (g_{1i}, y_{1j}) &= 0 = (g_{1i};f_{1j})_3 + (g_{1i};g_{1j})_4\end{aligned} \quad (109a)$$

and

$$\begin{aligned}(f_{2i}, x_{2j}) &= (f_{2i};f_{2j})_1 + (f_{2i};g_{2j})_2 \\ (g_{2i}, x_{2j}) &= (g_{2i};f_{2j})_3 + (g_{2i};g_{2j})_4\end{aligned} \quad (109b)$$

From the form of these equations, it can be seen that we simply have two systems of equations (corresponding to the value of $j = 1$ or 2) for the functions f_{ji} and g_{ji}. Consider now the new variational function

$$H = \langle Q;N\rangle + \lambda(Q, X) + \mu(N, Y) \quad (110)$$

where λ and μ are two Lagrangian multipliers. Independent variation with respect to Q and N yields the equations

$$2L_1(Q) + 2L_2(N) + \lambda X = 0$$
$$2L_3(Q) + 2L_4(N) + \mu Y = 0$$

which are equivalent to (99) provided we set $\lambda = \mu = -2$.

Applying these results to the system of equations (47) we introduce the variational functions

$$H_1 = \langle f_{1i}; g_{1j} \rangle + \lambda_1(f_{1i}, x_{1j}) + \mu_1(g_{1i}, y_{1j}) \tag{111a}$$
$$H_2 = \langle f_{2i}; g_{2j} \rangle + \lambda_2(f_{2i}, x_{2j}) + \mu_2(g_{2i}, y_{2j}) \tag{111b}$$

which, on variation with respect to f_{1i} and g_{1i}, and f_{2i} and g_{2i}, respectively, lead to the system of equations (47) provided we set $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = -2$.

a. Solution of the Generalized Variational Principle

The generalized variational principle (108) is again solved by an application of the Ritz method. For this purpose we expand the functions g_{1j} and g_{2j} in a power series in $y = h\nu/kT$, in addition to the expansion of the f's carried out previously (cf. 75).

$$g_{1j} = \sum_s c_{js} y^s u_j \tag{112a}$$

$$g_{2j} = \sum_s d_{js} y^s u_j. \tag{112b}$$

It should be noted that there are three independent directions of the lattice waves in general, and it is therefore not permissible to omit the index j from the quantities c_{js} and d_{js}, which are different for the various directions of the lattice polarization ($j = 1$ for longitudinal and $j = 2, 3$ for transverse waves). u_j are the components of velocity of the lattice waves. The coefficients a, b, c and d are then determined by the Ritz variational principle subject to the subsidiary conditions (109).

With the help of the expansions (75) and (112) the functions to be maximized can then be written as

$$\langle f_{1i}; g_{1j} \rangle = \sum_{r,s} (a_r a_s d_{rs}{}^{ij} + 2a_r c_{js} g_{rs}{}^{ij} + c_{is} c_{js} h_{rs}{}^{ij}) \tag{113a}$$

$$\langle f_{2i}; g_{2j} \rangle = \sum_{r,s} (b_r b_s d_{rs}{}^{ij} + 2b_r c_{js} g_{rs}{}^{ij} + d_{is} d_{js} h_{rs}{}^{ij}) \tag{113b}$$

where the coefficients d_{rs}^{ij}, g_{rs}^{ij} and h_{rs}^{ij} are given by

$$d_{rs}^{ij} = (\mathfrak{x}^r k_i;\ \mathfrak{x}^s k_j)_1 = d_{sr}^{ji} \tag{114a}$$

$$g_{rs}^{ij} = (\mathfrak{x}^r k_i;\ y^s u_j)_2 = (y^r u_i;\ \mathfrak{x}^s k_j)_3 \tag{114b}$$

$$h_{rs}^{ij} = (y^r u_i;\ y^s u_j)_4 = h_{sr}^{ji}. \tag{114c}$$

The subsidiary conditions (109) can also be transformed into linear equations in the unknowns a, b, c, and d and yield

$$\begin{aligned} \sum_r \alpha_r^{ij} a_r &= \sum_{r,s} (a_r a_s f_{rs}^{ij} + a_r c_{js} g_{rs}^{ij}) \\ \sum_r \gamma_r^{ij} c_{jr} &= \sum_{r,s} (c_{ir} a_s g_{sr}^{ij} + c_{ir} c_{js} h_{rs}^{ij}) \end{aligned} \tag{115a}$$

$$\begin{aligned} \sum_r \beta_r^{ij} b_r &= \sum_{r,s} (b_r b_s f_{rs}^{ij} + b_r d_{js} g_{rs}^{ij}) \\ \sum_r \delta_r^{ij} d_{jr} &= \sum_{r,s} (d_{ir} b_s g_{sr}^{ij} + d_{ir} d_{js} h_{rs}^{ij}) \end{aligned} \tag{115b}$$

where, in addition to the α_r and β_r defined previously (79), we also define

$$\gamma_r^{ij} = (u_i,\ y_{1j} y^r), \qquad \delta_r^{ij} = (u_i,\ y_{2j} y^r). \tag{116}$$

The variational functions H_l ($l = 1, 2$) (111) then become

$$H_1^{ij} = \sum_{r,s} (a_r a_s d_{rs}^{ij} + 2a_r c_{js}^{ij} g_{rs}^{ij} + c_{ir} c_{js} h_{rs}^{ij}) + \lambda_1 \sum_r \alpha_r^{ij} a_r + \mu_1 \sum_r \gamma_r^{ij} c_{jr} \tag{117a}$$

$$H_2^{ij} = \sum_{r,s} (b_r b_s d_{rs}^{ij} + 2b_r c_{js} g_{rs}^{ij} + d_{ir} d_{js} h_{rs}^{ij}) + \lambda_2 \sum_r \beta_r^{ij} b_r + \mu_2 \sum_r \delta_r^{ij} d_{jr}. \tag{117b}$$

Variation with respect to the quantities a_r, b_r, c_{jr} and d_{jr} then results again in a system of linear equations for these unknowns, where the Lagrangian multipliers λ_l and μ_l ($l = 1, 2$) are again found by multiplying with a_r etc., respectively, and comparing with the subsidiary conditions (109). The final system of equations is thus given by

$$\sum_s (f_{rs}^{ij} a_s + g_{rs}^{ij} c_{js}) - \alpha_r^{ij} = 0 \tag{118a}$$

$$\sum_s (g_{rs}^{ij} a_s + h_{rs}^{ij} c_{js}) - \gamma_r^{ij} = 0 \tag{118b}$$

$$\sum_s (f_{rs}{}^{ij}b_s + g_{rs}{}^{ij}d_{js}) - \beta_r{}^{ij} = 0 \tag{118c}$$

$$\sum_s (g_{rs}{}^{ij}b_s + h_{rs}{}^{ij}d_{js}) - \delta_r{}^{ij} = 0. \tag{118d}$$

The distribution functions f_l and g_l can again be expressed in terms of infinite determinants. Let D denote[4] the determinant obtained from the matrix

$$D_{rs} = \begin{pmatrix} d_{rs} & g_{rs} \\ g_{rs} & h_{rs} \end{pmatrix} \tag{119}$$

and $D_{x\alpha\gamma}$ and $D_{y\beta\delta}$ be the determinants obtained from D by adding a column of $\alpha\gamma$ or $\beta\delta$ and a row consisting of $\mathfrak{x}^r k_i$ or $y^r u_i$ in the appropriate places. The other elements and the last diagonal element are zero. Then it follows from (118), (75) and (112) and the property of adding determinants that the distribution functions f_{1i} and g_{1j} ($l = 1, 2$) are given by

$$f_{1i} = a_r \mathfrak{x}^r k_1 = D_{x\alpha\gamma}/D \tag{120a}$$

$$f_{2i} = b_r \mathfrak{x}^r k_i = D_{x\beta\delta}/D \tag{120b}$$

$$g_{1j} = c_{jr} \mathfrak{x}^r u_j = D_{y\alpha\gamma}/D \tag{120c}$$

$$g_{2j} = d_{jr} \mathfrak{x}^r u_j = D_{y\beta\delta}/D. \tag{120d}$$

In the absence of a lattice distribution function (and the corresponding interactions characterized by the operators L_i ($i = 2, 3$ and 4)) this, of course, reduces to the previous result (84).

b. Entropy Production

The connection between the variational function $\langle Q; N \rangle$ and entropy production can again easily be demonstrated. The transport equation for the phonon distribution function can be written in general form as (*cf.* 39)

$$\partial n/\partial t]_{\text{coll.}} = u \cdot \nabla r + \partial n/\partial t. \tag{121}$$

Multiplying this equation by $\ln n$ and *subtracting* from it the corresponding one for which n has been replaced by $n + 1$ (in accordance with the Bose-Einstein statistics) we obtain after integrating over $d\mathbf{q}$

$$\partial S_p/\partial t + \operatorname{div} \mathbf{s}_p = -k \int [\ln n - \ln (n + 1)](\partial n/\partial t)_{\text{coll.}} d\mathbf{q} \tag{122}$$

where S_p and $\mathbf{s}_p$ are the entropy density and entropy current density of the phonons, respectively, and defined by

[4] To simplify the notation, the superscripts i and j are being omitted in (119) and similar expressions.

$$S_p = -k \int [n \ln n - (n+1) \ln (n+1)] d\mathbf{q} \tag{123a}$$

$$\mathbf{s}_p = -k \int [n \ln n - (n+1) \ln (n+1)] \mathbf{u} d\mathbf{q}. \tag{123b}$$

For the corresponding entropy density and entropy current density of the carriers we again use the expressions (87), so that the total entropy density and entropy current density are given by their sums

$$S = S_e + S_p \qquad \text{and} \qquad \mathbf{s} = \mathbf{s}_e + \mathbf{s}_p. \tag{124}$$

If we now use the expansions for f and n, (22) and (44) and limit ourselves again to terms linear in Q and N, both conservation laws can be written as

$$\partial S/\partial t + \operatorname{div} \mathbf{s} = -1/T \int [Q(\mathbf{k}) - (\epsilon - \zeta)][L_1(Q) + L_2(N)] d\mathbf{k}$$
$$- 1/T \int [N(\mathbf{q}) - h\nu][L_3(Q) + L_4(N)] d\mathbf{q}$$

where we have made use of the operator equations for these functions (47)

$$X = L_1(Q) + L_2(N) \tag{125a}$$

$$Y = L_3(Q) + L_4(N) \tag{125b}$$

to simplify the right-hand side. As the energy remains constant during the type of collision considered here, the terms proportional to $\epsilon - \zeta$ and $h\nu$ give no contribution to the entropy production, so that we have finally

$$\partial S/\partial t + \operatorname{div} \mathbf{s} = -1/T \langle Q; F \rangle \tag{126}$$

which is (apart from the factor $1/T$) the variational function $\langle Q; F \rangle$ (105) and thus demonstrates independently that $\langle Q; F \rangle$ is a positive definite function as shown earlier.

c. Effect of the Magnetic Field

As has been shown previously, the presence of the magnetic field introduces an extra term into the transport equations characterized by the operator L' defined by (87) which obeys the symmetry relations (94). The appropriate transport equations in operator form (*cf.* 53) can be written as

$$X^+ = L_1(Q^+) + L_2(N) + L'(Q^+) \tag{127a}$$

$$Y = L_3(Q^+) + L_4(N). \tag{127b}$$

Analogous to (110) let us now define a variational function H by

$$H = \langle Q^-, Q^+; N\rangle + \lambda(Q^+, X) + \mu(N, Y) \tag{128}$$

where the inner product $\langle Q^-Q^+; N\rangle$ is given by

$$\langle Q^-Q^+; N\rangle = (Q^-; Q^+)_1 + (Q^-; N)_2 + (N; Q^+)_3 + (N; N)_4 + \tfrac{1}{2}(Q^-; Q^+)' + \tfrac{1}{2}(Q^+; Q^-)'. \tag{129}$$

The first four terms do not involve the magnetic field, and hence it is immaterial whether we use the + or − sign in the functions Q. Furthermore, on setting $Q^- = Q^+$, the last two terms in (129) disappear on account of (32) and (128), and thus reduce to (105) in the absence of the magnetic field.

Let Q^+ and N be two functions satisfying the integral equations (127) and Q^- and N the ones obtained on reversing the direction of the field H. Then on forming the scalar products (Q, X) and (N, Y), one obtains the following subsidiary conditions

$$(Q^-, X) = (Q^-; Q^+)_1 + (Q^-; N)_2 + (Q^-; Q^+)' \tag{130a}$$

$$(N, Y) = (N; Q^+)_3 + (N; N)_4. \tag{130b}$$

Independent variation of H with respect to the functions δQ^- and δN then yields the equations

$$(\delta Q^-; Q^+)_1 + (\delta Q^-; N)_2 + 1/2(\delta Q^-; Q^+)' + 1/2(Q^+; \delta Q^-)' + \lambda(\delta Q^-, X) = 0 \tag{131a}$$

$$(Q^-; \delta N)_2 + (\delta N, Q^+)_3 + (\delta N; N)_4 + (N; \delta N)_4 + \mu(\delta N, Y) = 0 \tag{131b}$$

which are satisfied by (127) on eliminating the Lagrangian multipliers λ and μ through the subsidiary conditions (120), which for this case are seen to be equal to $\lambda = \mu = -1$. (The difference from the previous result is due to the fact that we now consider Q^- and Q^+ to be distinct functions.)

This result can now be applied to the system of equations (53) and their conjugate ones by replacing $Q^\pm$ by $f_{1i}^\pm$ or $f_{2i}^\pm$, and N by g_{1j} or g_{2j}. The appropriate variational functions are given by

$$H_1 = \langle f_{1i}^-, f_{1i}^+; g_{1j}\rangle - (f_{1i}^- x_{1j}^+) - (g_{1i}, y_{1j}) \tag{132a}$$

$$H_2 = \langle f_{2i}^-, f_{2i}^+; g_{2j}\rangle - (f_{2i}^- x_{2j}^+) - (g_{2i}, y_{2j}). \tag{132b}$$

The solution of the variational principle is again achieved by the Ritz method, where the functions $f_{1i}^\pm$ and $f_{2i}^\pm$ are expanded according to (97), and coefficients d, g and h are defined by (114), except that the

coefficients $d_{rs}^{\pm}$ consist now of two terms according to (98) due to an additional term representing the magnetic field. The final system of equations is of the type (118), where, however, we have a different system of equations depending whether the + or − sign has been used for the quantities $a^{\pm}$ and $b^{\pm}$ (and coefficients d, $\alpha^{\pm}$ and $\beta^{\pm}$), depending on the direction of the magnetic field.

3. Variational Principle for Interband Scattering

In the case of interband scattering, we have seen that the inner product $(F;G)$ is defined as a sum of terms of the form $(F^m;G^n)$, where m and n indicate the bands. If, in addition to that inner product (*cf.* 58 and 59)

$$(F;G) = \sum_{m,n} (F^m;G^n) = \sum_{m,n} \int F^m L_{mn}(G^n) d\mathbf{k}_m \tag{133}$$

we also define the product

$$(F,X) = \sum_{m} (F^m,X^m) = \sum_{m} \int F^m X^m d\mathbf{k}_m \tag{134}$$

it is easily seen that the extremal function of the problem is given by (133) subject to the subsidiary condition

$$(F,X) = \sum_{m} (F^m,X^m) = \sum_{m,n} (F^m;F^n) = (F;F). \tag{135}$$

As before, we can again define a new variational function by

$$H = \sum_{m,n} (F^m;F^n) + \lambda \sum_{m} (F^m,X^m) \tag{136}$$

which, on variation with respect to δF^m leads to the system of operator equations

$$\sum_{n} L_{mn}(F^n) = X^m \tag{137}$$

where the Lagrangian multiplier is found to be equal to $\lambda = -2$ by comparison with the subsidiary conditions (135). Applying this to our system of equations (55) we again obtain variational functions H_1 and H_2 of the form (74), the only difference being that now each of the terms $(f;f)$ and (f,x) appears as a sum over bands.

a. Solution of the Variational Principle

The variational principle can again be solved by the Ritz method. For this purpose we expand the functions f_{1i}^m and f_{2i}^m as an infinite

power series in ϵ_m (or $\mathfrak{x}_m$)

$$f_{1i}^{m} = k_i \sum_r a_{mn}(\epsilon_m)^r \tag{138a}$$

$$f_{2i}^{m} = k_i \sum_r b_{mn}(\epsilon_m)^r \tag{138b}$$

where now the unknown coefficients a_{mr} and b_{mr} and the energy ϵ_m also depend on the band under consideration.

With the help of the expansion for f the inner products $(f_{1i}; f_{1j})$ and $(f_{2i}; f_{2j})$ become

$$(f_{1i}; f_{1j}) = \sum_{m,r} \sum_{n,s} a_{mr} a_{ns} d_{mr,ns}^{ij} \tag{139a}$$

$$(f_{2i}; f_{2j}) = \sum_{m,r} \sum_{n,s} b_{mr} b_{ns} d_{mr,ns}^{ij} \tag{139b}$$

where we have introduced the notation

$$d_{mr,ns}^{ij} = (k_i \epsilon_m^r; k_i \epsilon_n^s) = d_{ns,mr}^{ji} \tag{140}$$

while the subsidiary conditions become

$$\sum_{m,r} \sum_{n,s} a_{mr} a_{ns} d_{mr,ns}^{ij} = \sum_{mr} a_{mr} \alpha_{mr}^{ij} \tag{141a}$$

$$\sum_{m,r} \sum_{n,s} b_{mr} b_{ns} d_{mr,ns}^{ij} = \sum_{mr} b_{mr} \beta_{mr}^{ij} \tag{141b}$$

where we have defined

$$\alpha_{mr}^{ij} = (k_i \epsilon_m^r, X_{1j}^m) \tag{142a}$$

$$\beta_{mr}^{ij} = (k_i \epsilon_m^r, X_{2j}^m). \tag{142b}$$

The variational functions H_l $(l = 1, 2)$ are thus given by

$$H_1 = \sum_{mr} \sum_{ns} a_{mr} a_{ns} d_{mr,ns}^{ij} + \lambda_1 \sum_{mr} a_{mr} \alpha_{mr}^{ij} \tag{143a}$$

$$H_2 = \sum_{mr} \sum_{ns} b_{mr} b_{ns} d_{mr,ns}^{ij} + \lambda_2 \sum_{mr} b_{mr} \beta_{mr}^{ij}. \tag{143b}$$

To find the values of the parameters we now vary these equations with respect to a_{mr} and b_{mr}. From the conditions for an extremum we obtain the system of equations

$$2 \sum_{ns} a_{ns} d_{mr,ns}^{ij} + \lambda_1 \alpha_{mr}^{ij} = 0 \tag{144a}$$

$$2 \sum_{ns} a_{ns} d_{mr,ns}^{ij} + \lambda_2 \beta_{mr}^{ij} = 0. \tag{144b}$$

The unknown Lagrangian multipliers λ_1 and λ_2 are found on multiplying these equations by a_{mr} and b_{mr}, respectively, summing over m and n

$$2\sum_{mr}\sum_{ns} a_{mr}a_{ns}d_{mr,ns}{}^{ij} + \lambda_1 \sum_{mr} a_{mr}\alpha_{mr}{}^{ij} = 0$$

$$2\sum_{mr}\sum_{ns} b_{mr}b_{ns}d_{mr,ns}{}^{ij} + \lambda_2 \sum_{mr} b_{mr}\beta_{mr}{}^{ij} = 0$$

and comparing with the subsidiary conditions (141).

Thus from (144) we get two systems of linear equations for the unknowns a_{ns} and b_{ns} which satisfy the stationary condition and the subsidiary conditions (141)

$$\alpha_{mr}{}^{ij} = \sum_{ns} a_{ns}d_{mr,ns}{}^{ij} \tag{145a}$$

$$\beta_{mr}{}^{ij} = \sum_{ns} b_{ns}d_{mr,ns}{}^{ij}. \tag{145b}$$

We note that this result is analogous to our previous result (82) for the unknowns a_r and b_r, except that now we also have to carry out summations over the various bands. The two distribution functions f_{1i} and f_{2i} can again be obtained in terms of infinite determinants. Let D denote the determinant obtained from the matrix $d_{mr,ns}$ where the elements are grouped according to increasing m (and n) for a given r (and s). Thus, for example, the determinant corresponding to $m = n = r = s = 0, 1$ is given by

$$D = \begin{vmatrix} d_{00,00} & d_{00,10} & d_{00,01} & d_{00,11} \\ d_{10,00} & d_{10,10} & d_{10,01} & d_{10,11} \\ d_{01,00} & d_{01,10} & d_{01,01} & d_{01,11} \\ d_{11,00} & d_{11,10} & d_{11,01} & d_{11,11} \end{vmatrix}. \tag{146}$$

Furthermore, let $D_{m\alpha}$ and $D_{m\beta}$ be the determinants obtained from D by bordering it with a column of α_{ns} and β_{ns}, respectively, and a row of $k_i(\epsilon_m)^r$ in the appropriate places, the others and the last diagonal element being zero. Thus, for example, $D_{0\beta}$ corresponding to (146) is given by

$$D_{0\beta} = \begin{vmatrix} d_{00,00} & d_{00,10} & d_{00,01} & d_{00,11} & \beta_{00} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ d_{11,00} & d_{11,10} & d_{11,01} & d_{11,11} & \beta_{11} \\ k_i\epsilon_o{}^0 & 0 & k_i\epsilon_o{}^1 & 0 & 0 \end{vmatrix}. \tag{147}$$

Then, it follows that the functions $f_{1i}{}^m$ and $f_{2i}{}^m$ are given by

$$f_{1i}{}^m = -D_m{}^{ij}/D^{ij} \tag{148a}$$

$$f_{2i} = -D_m{}^{ij}/D^{ij}. \tag{148b}$$

b. Effect of a Magnetic Field

In the presence of an external magnetic field the operator equations (55) have to be replaced by (60) which have an additional term $L_m'(F^m)$ describing the effect of the magnetic field. It has been noticed that this operator only has a free index describing the effect of the magnetic field on the band in question. Hence, let us define the product $(F; F)'$

$$(F; F)' = \sum_m (F^m; F^m)' = \sum_m [F^m L_m'(F^m)] \tag{149}$$

which, on account of the symmetry properties of L_m', satisfies the relation

$$(F^-; G^+)' = \sum_m (F^{m-}; G^{m+})' = \sum_m (G^{m+}; F^{m-})' = (G^+; F^-)'. \tag{150}$$

In accordance with the procedure of Section 1, we take for our variational function H the combination

$$H = (F^-; F^+) + (F^-; F^+)' + \lambda(F^-, X) \tag{151}$$

where $(F^-; F^+)$ is the inner product defined by (133) and $(F^-; F^+)'$ the one by (149). The $+$ and $-$ signs again indicate the appropriate solutions depending on the direction of the field. Independent variation with respect to F^m then results in the operator equation

$$X^{m+} = \sum_n L_{mn}(F^{n+}) + L_m'(F^{n-}) \tag{152}$$

for the distribution function, where the Lagrangian multiplier is found again to be $\lambda = -2$ in order to satisfy the subsidiary conditions

$$\sum_m (F^{m-}, X^{m+}) = \sum_{mn} (F^{m-}; F^{n+}) + \sum_m (F^{m-}; F^{m+})'. \tag{153}$$

Applying this result to the system of equations (60), we now have for our new variational function H_l ($l = 1, 2$) expressions of the form (96), the only difference being that now each of the inner products has to be summed over the various bands in accordance with the definitions (133, 134 and 149). The solution is again obtained by the Ritz method, where now the functions $f_{1i}{}^{m\pm}$ and $f_{2i}{}^m$ are expanded according to

$$f_{1i}{}^{m\pm} = k_i \sum_r a_{mr}{}^{\pm}(\epsilon_m)^r \tag{154a}$$

$$f_{2i}{}^{\pm} = k_i \sum_r b_{mr}{}^{\pm}(\epsilon_m)^r \tag{154b}$$

and the new coefficients $d_{mr,ns}$ are defined by

$$d_{mr,ns}{}^{ij} = (k_i\epsilon_m{}^r; k_j\epsilon_r{}^s) + (k_i\epsilon_m{}^r; k_j\epsilon_r{}^s)'. \tag{155}$$

The rest of the calculation proceeds as before, and we finally obtain two systems of equations of the type (145) for the unknown coefficients $a_{ns}^{\pm}$ and $b_{ns}^{\pm}$, respectively, depending whether the + or − sign has been used for the unknowns a and b and the coefficients d, α and β. The distribution functions f can again be expressed in terms of infinite determinants, where again one has to differentiate between the + and − sign characteristic of the two solutions.

c. Presence of Lattice Vibrations

The carrier distribution function F^m depends on the band in question, while the lattice distribution function G does not. In view of this, it is necessary to differentiate between operators which operate on F^m and those which operate on G, and define the corresponding inner products accordingly, as can be seen from (64 and 65). Formally, however, we may expect that our previous results will still apply, provided that we take account of the appropriate definitions of the inner products.

Let us take then, for our variational function H

$$H = \langle Q; N\rangle + \lambda(Q, X) + \mu(N, Y) \tag{156}$$

where the bracket $\langle Q; N\rangle$ denotes the linear combination

$$\langle Q; N\rangle = (Q; Q)_1 + (Q; N)_2' + (N; Q)_3 + (N; N)_4' \tag{157}$$

whose components have been defined by (64). Furthermore, the products (Q, X) and (N, Y) are given, respectively, by

$$(Q, X) = \sum_m (Q^m, X^m) = \sum_m \int Q^m X^m d\mathbf{k}_m \tag{158a}$$

$$(N, Y) = \int NY d\mathbf{q}, \tag{158b}$$

only the first of which contains the dependence on the band. λ and μ are, of course, again Lagrangian multipliers. Independent variation of H with respect to Q^m and N yields the equations

$$\sum_n L_{mn}^{1}(Q^n) + \sum_n L_{mn}^{2}(N) = X^m \tag{159a}$$

$$\sum_{mn} L_{mn}^{3}(Q^n) + \sum_{mn} L_{mn}^{4}(N) = Y \tag{159b}$$

where the Lagrangian multipliers λ and μ are found to be equal to $\lambda = \mu = -2$ by comparison with the subsidiary conditions

$$(Q; Q)_1 + (Q; N)_2' = (Q, X) \tag{160a}$$

$$(N; Q)_3 + (N; N)_4' = (N, Y). \tag{160b}$$

Applying our result to the system of equations (61), we use the variational functions of the form (111), where the various terms are to be interpreted in terms of our definitions for inner products, such as (64) and (158). Variation with respect to f_{1i}^{m} and g_{1i} and f_{2i}^{m} and g_{2i}, respectively, then leads to the system of equations (61), where the Lagrangian multipliers have been eliminated with the help of the subsidiary conditions (109). In order to solve the variational principle we expand the functions f_{1i}^{m} and f_{2i}^{m} according to (138) and g_{1i} and g_{2i} according to (54). With the help of these expansions the functions to be maximized can then be written as

$$\langle f_{1i}; g_{1j}\rangle = \sum_{mr}\sum_{ns}\left[a_{mr}b_{ns}d_{mr,ns}{}^{ij}+2a_{mr}c_{js}g_{mr,ns}{}^{ij}+c_{ir}c_{js}h_{mr,ns}{}^{ij}\right] \quad (161a)$$

$$\langle f_{2i}; g_{2j}\rangle = \sum_{mr}\sum_{ns}\left[b_{mr}b_{ns}d_{mr,ns}{}^{ij}+2b_{mr}d_{js}g_{mr,ns}{}^{ij}+d_{ir}d_{js}h_{mr,ns}{}^{ij}\right] \quad (161b)$$

where the coefficients $d_{mr,ns}$ are given by

$$d_{mr,ns}{}^{ij} = (\epsilon_m{}^r k_i; \epsilon_n{}^1 k_j)_{mn}{}^1 = d_{ns,mr}{}^{ij} \quad (162a)$$

$$g_{mr,ns}{}^{ij} = (\epsilon_m{}^r k_i; y^s u_j)_{mn}{}^2 = (y^r u_i; \epsilon_n{}^s k_j)_{mn}{}^2 \quad (162b)$$

$$h_{mr,ns}{}^{ij} = (y^r u_i; y^s u_j)_{mn}{}^4. \quad (162c)$$

The subsidiary conditions can also be transformed into linear equations in the unknowns and are given by

$$\sum_{m,r}\alpha_{mr}{}^{ij}a_{mr} = \sum_{mr}\sum_{ns}\left(a_{mr}a_{ns}d_{mr,ns}{}^{ij} + a_{mr}c_{js}g_{mr,ns}{}^{ij}\right) \quad (163a)$$

$$\sum_{r}\gamma_r{}^{ij}c_{jr} = \sum_{mr}\sum_{ns}\left(c_{ir}a_{ns}g_{ns,mr}{}^{ij} + c_{ir}c_{js}h_{mr,ns}{}^{ij}\right) \quad (163b)$$

$$\sum_{mr}\beta_{mr}{}^{ij}b_{mr} = \sum_{mr}\sum_{ns}\left(b_{mr}b_{ns}d_{mr,ns}{}^{ij} + b_{mr}c_{js}g_{mr,ns}{}^{ij}\right) \quad (163c)$$

$$\sum_{r}\delta_r{}^{ij}c_{jr} = \sum_{mr}\sum_{ns}\left(d_{ir}b_{ns}g_{ns,mr}{}^{ij} + d_{ir}d_{js}h_{mr,ns}{}^{ij}\right) \quad (163d)$$

where the α_{mr} and β_{mr} are defined by (142) and γ_r and δ_r by (116) in accordance with (158).

The variational functions H_l ($l = 1, 2$) then become

$$H_1 = \langle f_{1i}; g_{1j}\rangle + \lambda_1 \sum \alpha_{mr}{}^{ij}a_{mr} + \mu_1 \sum \gamma_r{}^{ij}c_{jr} \quad (164a)$$

$$H_2 = \langle f_{2i}; g_{2j}\rangle + \lambda_2 \sum \beta_{mr}{}^{ij}b_{mr} + \mu_2 \sum \delta_r{}^{ij}c_{jr}. \quad (164b)$$

Variation with respect to the quantities a_{mr}, b_{mr}, c_{jr} and d_{jr} then results again in a system of linear equations for these unknowns, which after

elimination of the Lagrangian multipliers finally results in the system of equations

$$\sum_{ns} (d_{mr,ns}{}^{ij}a_{ns} + g_{mr,ns}{}^{ij}c_{js}) - \alpha_{mr}{}^{ij} = 0 \tag{165a}$$

$$\sum_{m}\sum_{ns} (g_{mr,ns}{}^{ij}a_{ns} + h_{mr,ns}{}^{ij}c_{js}) - \gamma_r{}^{ij} = 0 \tag{165b}$$

$$\sum_{ns} (d_{mr,ns}{}^{ij}b_{ns} + g_{mr,ns}{}^{ij}d_{js}) - \beta_{mr}{}^{ij} = 0 \tag{165c}$$

$$\sum_{m}\sum_{ns} (g_{mr,ns}b_{ns} + h_{mr,ns}d_{js}) - \delta_r{}^{ij} = 0. \tag{165d}$$

The distribution function $f_l{}^m$ and g_l ($l = 1, 2$) can again be expressed in terms of infinite determinants. For this purpose, we define D to be the determinant obtained from the matrix

$$D_{mr,ns} = \begin{pmatrix} d_{mr,ns} & g_{mr,ns} \\ g_{mr,ns} & h_{mr,ns} \end{pmatrix} \tag{166}$$

where the elements have been ordered according to the convention leading to (146). Furthermore, we define $D_{\epsilon m\alpha\gamma}$ and $D_{\epsilon m\beta\delta}$ to be the determinants obtained from D by adding a column of $\alpha_{mr}\gamma_r$ or $\beta_{mr}\delta_r$ and a row consisting of $\epsilon_m{}^r k_i$ in the appropriate places, the others and the last diagonal element being zero. Also let $D_{y\alpha\gamma}$ and $D_{y\beta\delta}$ be again the determinants obtained from D by bordering it with a column of $\alpha_{mr}\gamma_r$ or $\beta_{mr}\delta_r$ and a row of $y^r u_i$, the last diagonal element being zero. Then it follows from the properties of determinants that the distribution functions f^m and g are given by

$$f_{1i}{}^m = \sum_r a_{mr}\epsilon_m{}^r k_i = D_{\epsilon m\alpha\gamma}/D \tag{167a}$$

$$f_{2i}{}^m = \sum_r b_{mr}\epsilon_m{}^r k_i = D_{\epsilon m\beta\delta}/D \tag{167b}$$

$$g_{1y} = \sum_r c_{jr}y^r u_j = D_{y\alpha\gamma}/D \tag{167c}$$

$$g_{2y} = \sum_r d_{jr}y^r u_j = D_{y\beta\delta}/D. \tag{167d}$$

If, in addition to the lattice distribution function g, we also have an external magnetic field, the appropriate variational principle is given by one of the form (132), where the various terms have to be interpreted in accordance with previous results. For example, the products $(f_{1i}{}^-, x_{ij}{}^+)$ and $(f_{2i}{}^-, x_{2j}{}^+)$ are to be considered as sums over the various bands as in (134) with the $+$ and $-$ signs indicating the direction of the field. The inner product $\langle f^- f^+; g\rangle$ is of the form (129) where, in addition to

the terms defined by (122), we also have to include those characteristic of the magnetic field and given by (149). Independent variation with respect to $f_1^{m\pm}$ and g_{1j} then leads to the appropriate equations. The solution of this more complicated variational principle is again achieved by the Ritz method, where the functions $f_1^{m\pm}$ and $f_2^{m\pm}$ are expanded according to (154) and g_1 and g_2 according to (112). The coefficients d, g are defined as in (162) except that the coefficients $d_{mr,ns}$ consist now of two terms and are given by (155). The final system of equations is of the type (165), where, however, we now have two sets of equations depending on whether the + or − sign has been used for the quantities $a_{mr}^{\pm}$ and $b_{mr}^{\pm}$ (and coefficients) depending on the direction of the magnetic field.

This, then, concludes our discussion of the variational principle as applied to transport theory, where we have tried to discuss the various cases arising whether (*i*) an external magnetic field, (*ii*) non-equilibrium lattice distributions or (*iii*) interband scattering are present, singly or in combination. In the next section we shall apply our result to the evaluation of the various transport coefficients.

IV. TRANSPORT PHENOMENA

1. *General Formulation of Transport Coefficients*

The various transport coefficients can be obtained from the general expressions for the electric current density **J** and energy current density **W**. The former density is given by

$$J_i = -(e/8\pi^3)\int v_i f d\mathbf{k} \qquad (i = 1, 2, 3) \tag{168}$$

while the latter consists of two contributions, one due to the electronic energy density, and one due to the lattice current density, and is given by

$$W_i = (1/8\pi^3)\int v_i \epsilon f d\mathbf{k} + (1/8\pi^3)\int u_i h\nu_q n d\mathbf{q} \qquad (i = 1, 2, 3). \tag{169}$$

Here f and n denote again the carrier and lattice distribution functions for which we use their respective expansions (22) and (44). The integrals containing f_o or n_o will give no contributions to the current and energy current densities, while the terms containing Q and N can be separated into two parts according to (27) and (48). Of course, in the simple case where we neglect the deviation of the lattice distribution function from equilibrium, the latter will yield no contribution. Using these expansions we obtain for J_i and W_i (where $i = 1$ to 3 denote the components of **J** and **W**)

$$J_i = \sum_j \{K_{ij}^{(1)}[e^2 E_j + eT\partial/\partial x^j(\zeta/T)] + K_{ij}^{(2)}(e/T)\partial T/\partial x^j\} \tag{170}$$

and

$$W_i = -\sum_j \{K_{ij}^{(3)}[eE_j + T\partial/\partial x^j(\zeta/T)] + K_{ij}^{(4)}(i/T)\partial T/\partial x^j, \quad (171)$$

respectively, where we have defined the energy integrals

$$8\pi^3 K_{ij}^{(1)} = \int v_i f_{1j} \partial f_o/\partial \epsilon d\mathbf{k} = (f_{1j}, x_{1i}) \quad (172a)$$

$$8\pi^3 K_{ij}^{(2)} = \int v_i f_{2j} \partial f_o/\partial \epsilon d\mathbf{k} = (f_{2j}, x_{1i}) \quad (172b)$$

$$8\pi^3 K_{ij}^{(3)} = \int v_i \epsilon f_{1j} \partial f_o/\partial \epsilon d\mathbf{k} - \int u_i T g_{1j} \partial n_o/\partial T d\mathbf{q} = (f_{1j}, x_{2i}) + (g_{1j}, y_{2i}) \quad (172c)$$

$$8\pi^3 K_{ij}^{(4)} = \int v_i \epsilon f_{2j} \partial f_o/\partial \epsilon d\mathbf{k} - \int u_i T g_{2j} \partial n_o/\partial T d\mathbf{q} = (f_{2j}, x_{2i}) + (g_{2j}, y_{2i}) \quad (172d)$$

and where we have also made use of the definition (69). Hence it is seen that the energy integrals (172) are nothing else than the subsidiary conditions (109). The K's defined above are not independent, but satisfy the so-called Kelvin relations

$$K_{ij}^{(1)} = K_{ji}^{(1)}, \qquad K_{ij}^{(2)} = K_{ji}^{(3)}, \qquad K_{ij}^{(4)} = K_{ji}^{(4)}. \quad (173)$$

The proof of (173) can be carried out with the help of the symmetry relations (52) and the operator equations (47).

From the fact that the electric current density J_i can be written as

$$J_i = \sum_j \sigma_{ij} E_j \quad (174)$$

we deduce from (168) that the conductivity tensor σ_{ij} is given by

$$\sigma_{ij} = e^2 K_{ij}^{(1)}. \quad (175)$$

If we denote the reciprocal tensor of $K_{ij}^{(1)}$ by $\tilde{K}_{ij}^{(1)}$, then the electrical resistivity ρ_{ij} is given by

$$\rho_{ij} = (1/e^2)\tilde{K}_{ij}^{(1)}. \quad (176)$$

Equations (170) and (171) can be solved for the electric field and the heat current density

$$E_i = \sum_j (\rho_{ij} J_j - (1/e)\delta_{ij}\partial T/\partial x^j) - (1/e)\partial\zeta/\partial x^i \quad (177)$$

$$W_i = \sum_j (\Pi_{ij} J_j - \kappa_{ij}\partial T/\partial x^j) - (\zeta/T) J_i \quad (178)$$

where S_{ij} is the tensor of the Seebeck coefficient, Π_{ij} that of the Peltier coefficient, and κ_{ij} the heat conductivity tensor. Comparisons of (177) and (178) with (170) and (171) yields for these tensors

$$TS_{ij} = + (\zeta/e)\delta_{ij} + \sum_m \bar{K}_{im}^{(1)} K_{mj}^{(2)} \tag{179}$$

$$\Pi_{ij} = - (\zeta/e)\delta_{ij} - \sum_m K_{jm}^{(2)} \bar{K}_{mi}^{(1)} \tag{180}$$

$$T\kappa_{ij} = K_{ij}^{(4)} - \sum_{m,n} K_{im}^{(3)} \bar{K}_{mn}^{(1)} K_{nj}^{(2)} \tag{181}$$

where δ_{ij} denotes the Kroenicker delta, $\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ if $i \neq j$. Moreover, from the Kelvin relations (173), it also follows that

$$\Pi_{ij} = - TS_{ji} \tag{182}$$

in accordance with known results.

The expressions for the various transport coefficients (175, 176, 179–182) are quite general and do not depend on any particular model used. Their value depends only on the energy integrals which we shall evaluate for the various cases of interest by the Ritz variational principle.

2. Effect of Magnetic Field

We have seen that in the presence of an external magnetic field the transport equations acquire an extra term due to the interaction with the magnetic field, and that the distribution functions f_1 and f_2 have to be divided into two conjugate sets f_l^+ and f_l^- ($l = 1, 2$) corresponding to the direction of the field. The effect of this separation will be felt in the energy integrals which are now a function of the magnetic field $\mathbf{H}$. The electric current and heat current densities are still given by their respective expressions (170) and (171), but the energy integrals K_{ij} are now defined by

$$8\pi^3 K_{ij}^{(1)}(\mathbf{H}) = (f_{1j}^-, x_{1i}^+) \tag{183a}$$

$$8\pi^3 K_{ij}^{(2)}(\mathbf{H}) = (f_{2j}^-, x_{1i}^+) \tag{183b}$$

$$8\pi^3 K_{ij}^{(3)}(\mathbf{H}) = (f_{1j}^-, x_{2i}^+) + (g_{1j}, y_{2i}) \tag{183c}$$

$$8\pi^3 K_{ij}^{(4)}(\mathbf{H}) = (f_{2j}^-, x_{2i}^+) + (g_{2j}, y_{2i}) \tag{183d}$$

where the products (f^-, x^+) are of the type (69) with the appropriate direction of the magnetic field $\mathbf{H}$. From (183) we can obtain the conjugate set corresponding to $-\mathbf{H}$ by changing all $+$ to $-$ signs and vice

versa. It can be seen that the energy integrals (183) are again the subsidiary conditions (130) which differ from the previous ones (109) by having an extra term of the form $(f_l^-; f_l^+)'$ and two sets of the distribution function f_l $(l = 1, 2)$.

With the help of the symmetry relations (108) and (94) for inner products involving the operator L' (31) it can be shown that they satisfy the Kelvin relations (173), where now, however, the change of direction of the magnetic field must be taken into account. We thus obtain

$$\begin{aligned} K_{ij}^{(1)}(\mathbf{H}) &= K_{ji}^{(1)}(-\mathbf{H}); \qquad K_{ij}^{(2)}(\mathbf{H}) = K_{ji}^{(3)}(-\mathbf{H}) \\ K_{ij}^{(4)}(\mathbf{H}) &= K_{ji}^{(4)}(-\mathbf{H}). \end{aligned} \tag{184}$$

The various transport coefficients are still given by their appropriate expressions, (175, 176, 179–181), but where now the energy integrals defined by (183) must be used. From their symmetry relations under reversal of the magnetic field (184), one sees readily that the following relations hold among the various transport quantities

$$\begin{aligned} \sigma_{ij}(\mathbf{H}) &= \sigma_{ji}(-\mathbf{H}); \qquad K_{ij}(\mathbf{H}) = K_{ji}(-\mathbf{H}) \\ TS_{ij}(\mathbf{H}) &= -\Pi_{ji}(-\mathbf{H}). \end{aligned} \tag{185}$$

If one separates the above tensors into their symmetric and antisymmetric parts, one finds that the symmetric parts of these tensors are even functions of the magnetic field, while their antisymmetric parts are odd functions of the field, that is, they change sign on reversal of the field.

3. *Calculation of Energy Integrals*

We have seen in the previous sections that all thermoelectric coefficients can be expressed in terms of the energy integrals K_{ij}. Thus, te obtain expressions for these integrals, it is only necessary to calculato the various thermoelectric and magnetic coefficients. By applying an elegant method due to Enskog to the various cases, these can be obtained in terms of infinite determinants without the explicit solution of the transport equations.

a. Carrier Distribution Function

Let us first consider the simple case for which the deviation of the lattice distribution function from equilibrium has been neglected, that is, $n = n_o$. The appropriate energy integrals are then given by

$$8\pi^3 K_{ij}^1 = \int v_1 f_{1j}(\partial f_o/\partial \epsilon) d\mathbf{k} = (f_{1j}, x_{1i}) \tag{186a}$$

$$8\pi^3 K_{ij}^2 = \int v_1 f_{2j}(\partial f_o/\partial \epsilon) d\mathbf{k} = (f_{2j}, x_{1i}) \tag{186b}$$

$$8\pi^3 K_{ij}^3 = \int v_i \epsilon f_{1j} (\partial f_o/\partial \epsilon) d\mathbf{k} = (f_{1j}, x_{2i}) \tag{186c}$$

$$8\pi^3 K_{ij}^4 = \int v_i \epsilon f_{2j} (\partial f_o/\partial \epsilon) d\mathbf{k} = (f_{2j}, x_{2i}) \tag{186d}$$

where the distribution functions f_{1i} and f_{2i} are given by (184) and x_{1i} and x_{2i} are defined by (27). From these definitions and the ones for α_r and β_r (79), it is seen that the various energy integrals are just sums of products of the coefficients a_r and b_r and the integrals α_r and β_r. Thus, if we denote by $D_{\alpha\beta}$ the determinant obtained from $D = |d_{rs}|$ by adding a column of β_r and a row of α_r (with similar additions for the other types), we obtain for the energy integrals (186)

$$8\pi^3 K_{ij}^1 = \sum_r a_r \alpha_r^{ij} = -D_{\alpha\alpha}/D \tag{187a}$$

$$8\pi^3 K_{ij}^2 = \sum_r b_r \alpha_r^{ij} = -D_{\alpha\beta}/D \tag{187b}$$

$$8\pi^3 K_{ij}^3 = \sum_r a_r \beta_r^{ij} = -D_{\beta\alpha}/D \tag{187c}$$

$$8\pi^3 K_{ij}^4 = \sum_r b_r \beta_r^{ij} = -D_{\beta\beta}/D. \tag{187d}$$

The addition of an external magnetic field leaves that result unchanged in principle, except that the matrix elements d_{rs} (77) have to be replaced by the corresponding ones including the effect of the field (98), and + or − signs have to be used for the coefficients $\alpha_r^{\pm}$ and $\beta_r^{\pm}$ depending on the direction of the field. In other words, we will now have two sets of energy integrals $K_{ij}(\mathbf{H})$ and $K_{ij}(-\mathbf{H})$ of the type

$$8\pi^3 K_{ij}^2(\mathbf{H}) = -D_{\alpha^-\beta^+}/D; \qquad 8\pi^3 K_{ij}^2(-\mathbf{H}) = -D_{\alpha^+\beta^-}/D \tag{188}$$

and similar ones for the other cases in (187).

b. Effect of Lattice Vibrations

In the general case where the deviation of the lattice distribution function from equilibrium is also taken into account, the energy integrals K_{ij} are given by (172), the distribution functions f_{li} and g_{li} ($l = 1, 2$) by (120), and the coefficients α_r, β_r, γ_r and δ_r by (79) and (116), respectively. It can be seen again that the energy integrals are just sums of products of the coefficients a_r, b_r, c_r and integrals α_r, β_r, γ_r and δ_r, and therefore can be expressed again in terms of infinite determinants. With the help of the defining equations and the properties

of adding of determinants we obtain for the energy integrals K_{ij} in this case

$$8\pi^3 K_{ij}^1 = \sum_r (a_r\alpha_r^{ij} + c_{jr}\gamma_r^{ij}) = -D_{\alpha\gamma\alpha\gamma}/D \tag{189a}$$

$$8\pi^3 K_{ij}^2 = \sum_r (b_r\alpha_r^{ij} + d_{jr}\gamma_r^{ij}) = -D_{\beta\delta\alpha\gamma}/D \tag{189b}$$

$$8\pi^3 K_{ij}^3 = \sum_r (a_r\beta_r^{ij} + c_{jr}\delta_r^{ij}) = -D_{\alpha\gamma\beta\delta}/D \tag{189c}$$

$$8\pi^3 K_{ij}^4 = \sum_r (b_r\beta_r^{ij} + d_{jr}\delta_r^{ij}) = -D_{\beta\delta\beta\delta}/D \tag{189d}$$

where $D_{\alpha\gamma\beta\delta}$ is the determinant obtained from D by a line consisting of $\alpha\gamma$ and a column of $\beta\delta$ with similar definitions for the other determinants. (It should be noted that in this case D is the determinant obtained from the matrix D_{rs} defined by (119).)

As in the simple case discussed above, the addition of an external magnetic field provides only minor modifications of that result. For the matrix elements d_{rs} we have to use the ones which include the effect of the field (98), and + and − signs have to be used for the quantities $\alpha_r^{\pm}$ and $\beta_r^{\pm}$, depending on the direction of the field. Of course, the integrals $\gamma_r^{\pm}$ and $\delta_r^{\pm}$ are not affected by the field and remain unchanged. The final expressions for the energy integrals are still of the form (189), with the modification introduced by (88) resulting in two sets of integrals $K_{ij}(\mathbf{H})$ and $K_{ij}(-\mathbf{H})$ as defined by (183), from which the various transport quantities are to be calculated in accordance with (175, 176, 179–181).

4. *Transport Coefficients for Interband Scattering*

a. *Without Taking Account of Lattice Vibrations*

In the case of interband scattering the electric and energy current densities may be considered to be given as sums of the appropriate expressions for each band. Thus

$$\mathbf{J} = \sum_m \mathbf{J}^m \tag{190}$$

where

$$J_i^m = -(e/8\pi^3)\int v_i f^m d\mathbf{k} = \sum_j \{K_{ij}^{m1}[e^2E_j + eT\partial/\partial x_j(\zeta/T)] + K_{ij}^{m2}(e/T)\partial T/\partial x_j\} \tag{191}$$

and

$$\mathbf{W} = \sum \mathbf{W}_\alpha^m \tag{192}$$

with

$$W_i^m = (1/8\pi^3)\int v_i e f^m d\mathbf{k} = -\sum_j \{K_{ij}^{m3}[eE_j + T\partial/\partial x_j(\zeta/T)] + K_{ij}^{m4}(1/T)\partial T/\partial x_j\}. \quad (193)$$

The K's are the energy integrals for the various bands and analogous to (186) defined by

$$8\pi^3 K_{ij}^{1m} = (f_{1j}^m, x_{1i}^m) \quad (194a)$$

$$8\pi^3 K_{ij}^{2m} = (f_{2j}^m, x_{1i}^m) \quad (194b)$$

$$8\pi^3 K_{ij}^{3m} = (f_{1j}^m, x_{2i}^m) \quad (194c)$$

$$8\pi^3 K_{ij}^{4m} = (f_{2j}^m, x_{2j}^m). \quad (194d)$$

As before, the indices m and n refer to the bands and i and j to the direction. From (190–193) it can be seen that the current and energy current densities can be written as

$$J_i = \sum_j \{K_{ij}^1[e^2E_j + eT(\partial/\partial x_j)(\zeta/T)] + K_{ij}^2(e/T)\partial T/\partial x_j\} \quad (195)$$

and

$$W_i = -\sum_j \{K_{ij}[eE_j + T(\partial/\partial x_j)(\zeta/T)] + K_{ij}(1/T)\partial T/\partial x_j\}, \quad (196)$$

respectively, where we introduced the notation

$$K_{ij}^u = \sum_m K_{ij}^{um} \qquad (u = 1, 2, 3 \text{ or } 4). \quad (197)$$

The expressions (195) and (196) are, therefore, of the same form as the corresponding ones in the standard theory (170) and (171), the only difference being the definition (197) for the energy integrals. From (195) and (196), all transport coefficients can now be obtained in the usual manner (see Sections 1 and 2).

The energy integrals can be obtained again in terms of infinite determinants by applying the results of Section III to this case. It has been shown there that the distribution functions f_{1i}^m and f_{2i}^m are given by (148), and the various K's are then obtained by forming the inner products of the f's with x_1 or x_2 in accordance with (197). Thus, denote by $D_{m\alpha\beta}^{ij}$ the determinant obtained from $D_{m\beta}^{ij}$ by replacing the row of $k_i\epsilon_m^r$ by α_{mr} in the appropriate places, the others and the last diagonal element being again zero. In other words, $D_{m\alpha\beta}^{ij}$ is of the same form as $D_{\alpha\beta}$ except that it has more rows and columns and some zeros in the last row. For example, $D_{m\alpha\beta}^{ij}$ obtained from $D_{o\beta}^{ij}$ (147) is given by

$$D_{\alpha\beta} = \begin{vmatrix} d_{00,00} & d_{00,10} & d_{00,01} & d_{00,11} & \beta_{00} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ d_{11,00} & d_{11,10} & d_{11,01} & d_{11,11} & \beta_{11} \\ \alpha_{00} & 0 & \alpha_{01} & 0 & 0 \end{vmatrix}. \tag{198}$$

We then obtain for the various $K_{ij}{}^{u}$'s

$$K_{ij}{}^{1m} = \sum_r a_{mr}\alpha_{mr} = -D_{m\alpha\alpha}{}^{ij}/D \tag{199a}$$

$$K_{ij}{}^{2m} = \sum_r b_{mr}\alpha_{mr} = -D_{m\alpha\beta}/D \tag{199b}$$

$$K_{ij}{}^{3m} = \sum_r a_{mr}\beta_{mr} = -D_{m\beta\alpha}/D \tag{199c}$$

$$K_{ij}{}^{4m} = \sum_r b_{mr}\beta_{mr} = -D_{m\beta\beta}/D. \tag{199d}$$

Summation over m then yields the $K_{ij}{}^{u}$ ($u = 1, 2, 3$ or 4) (197). Inspection of the determinants $D_{m\alpha\beta}$ such as (198) shows that the summation over m can be carried out directly in the determinants. Hence, if we define $D_{\alpha\beta}$ to be a determinant obtained from D by bordering it with a row of α_{mr} and a column of β_{mr} (defined by (142)), where the last diagonal element is again zero, the $K_{ij}{}^{u}$ are then given by

$$K_{ij}{}^{1} = -D_{\alpha\alpha}/D \tag{200a}$$

$$K_{ij}{}^{2} = -D_{\alpha\beta}/D \tag{200b}$$

$$K_{ij}{}^{3} = -D_{\beta\alpha}/D \tag{200c}$$

$$K_{ij}{}^{4} = -D_{\beta\beta}/D \tag{200d}$$

in accordance with our previous results (187), the only difference being that now we have more rows and columns corresponding to the different values of (mr, ns). The various transport coefficients are then obtained from (200) and the appropriate formulae.

The effect of an external magnetic field is taken into account by replacing the matrix elements $d_{mr,ns}$ by the ones including the effect of the field (155), and supplying $+$ or $-$ signs to the integrals $\alpha_{mr}{}^{\pm}$ and $\beta_{mr}{}^{\pm}$ depending on the direction of the field. As before, we will again have two sets of energy integrals $K_{ij}{}^{u}(\mathbf{H})$ and $K_{ij}{}^{u}(-\mathbf{H})$ of the type (188), the only difference being that each set is of the form (200).

b. Presence of Lattice Vibrations

In the case for which deviations from equilibrium of the lattice distribution function are also taken into account, the energy integrals

$K_{ij}{}^{um}$ for the individual bands are given analogous to (172) by

$$8\pi^3 K_{ij}{}^{1m} = (f_{1j}{}^m, x_{1i}{}^m) \tag{201a}$$

$$8\pi^3 K_{ij}{}^{2m} = (f_{2j}{}^m, x_{1i}{}^m) \tag{201b}$$

$$8\pi^3 K_{ij}{}^{3m} = (f_{1j}{}^m, x_{2i}{}^m) + (g_{1j}, y_{2i}) \tag{201c}$$

$$8\pi^3 K_{ij}{}^{4m} = (f_{2j}{}^m, x_{2i}{}^m) + (g_{2j}, y_{2i}). \tag{201d}$$

The distribution functions $f_{li}{}^m$ and $g_{lj}{}^m$ ($l = 1, 2$) are given by (167), where the appropriate integrals and matrix elements have been defined by (116) and (142). The various K's are then obtained from (167) by forming the inner products of sums of f's and g's with x and y. From (201), the various defining equations and the properties of addition of determinants, we then obtain for the energy integrals in this case

$$8\pi^3 K_{ij}{}^{1m} = \sum_r (a_{mr}\alpha_{mr}{}^{ij} + c_{jr}\gamma_r{}^{ij}) = -D_{m\alpha\gamma\alpha\gamma}/D \tag{202a}$$

$$8\pi^3 K_{ij}{}^{2m} = \sum_r (b_{mr}\alpha_{mr}{}^{ij} + d_{jr}\gamma_r{}^{ij}) = -D_{m\beta\delta\alpha\gamma}/D \tag{202b}$$

$$8\pi^3 K_{ij}{}^{3m} = \sum_r (a_{mr}\beta_{mr}{}^{ij} + c_{jr}\delta_r{}^{ij}) = -D_{m\alpha\gamma\beta\delta}/D \tag{202c}$$

$$8\pi^3 K_{ij}{}^{4m} = \sum_r (b_{mr}\beta_{mr}{}^{ij} + d_{jr}\delta_r{}^{ij}) = -D_{m\beta\delta\beta\delta}/D \tag{202d}$$

where the determinant $D_{m\beta\delta\alpha\gamma}$ is the sum of the determinants $D_{\epsilon m\alpha\gamma}$ and $D_{y\alpha\gamma}$ replacing, however, the row of $\epsilon_m{}^r k_i$ by β_{mr} or α_{mr} and the row of $y^r u_i$ by γ_r or δ_r in the appropriate places, the others and the last diagonal element being zero. Summation over m then yields the $K_{ij}{}^u$ for this case. It can be seen that this summation can be carried out again directly in the determinants. Thus, if we denote by $D_{\alpha\gamma\beta\delta}$ the determinant obtained from D (as given in (166)) by bordering it with a row of $\alpha_{mr}\gamma_r$ and a column of $\beta_{mr}\gamma_r$, where the last diagonal element is again zero, the $K_{ij}{}^u$ are given by the previous expressions (189), the difference being that we now have more rows and columns corresponding to the different values of (mr, ns) and that the various matrix elements, $d_{mr,ns}g_{mr,ns}$ are given by (162) and the integrals α_{mr}, β_{mr}, γ_r, δ_r by (116) and (142).

The modification introduced by an external magnetic field results again in two sets of energy integrals $K_{ij}{}^u(\mathbf{H})$ and $K_{ij}{}^u(-\mathbf{H})$ with the matrix elements $d_{mr,ns}$ given by (155) and $+$ or $-$ signs introduced in the quantities $\alpha_{mr}{}^{\pm}$ and $\beta_{mr}{}^{\pm}$ depending on the direction of the field. From these expressions for the energy integrals, the various transport quantities can again be calculated in accordance with their defining equations (175, 176, 179–181) as before.

We have thus been able to demonstrate how a variational principle can be formulated which is equivalent to the transport equations for carrier distribution functions, lattice vibrations, external magnetic field, interband scattering and combinations of all or any of these. General expressions for the various transport coefficients have been given in terms of energy integrals, which have been evaluated in terms of infinite determinants for any of the above cases. To evaluate the different determinants the coefficients d, etc. must be calculated for which purpose it is necessary to make definite assumptions about the form of the interaction and energy dependence. Work along these lines for the calculation of the transport coefficients for warped energy surfaces is in progress and has been reported elsewhere (12). In this paper we have only been interested to give a systematic discussion of the variational principle as applied to transport theory and its application to the calculation of the transport coefficients.

REFERENCES

(1) P. M. MORSE AND H. FESCHBACH, "Methods of Theoretical Physics," New York, McGraw-Hill Book Co., Inc., 1953, Part II, Chapter 9.
(2) A. H. WILSON, "The Theory of Metals," New York, Cambridge University Press, 1953.
(3) M. KOHLER, *Z. Physik*, Vol. 124, p. 772 (1948); *ibid.*, Vol. 125, p. 679 (1949).
(4) J. M. ZIMAN, *Can. J. Phys.*, Vol. 34, p. 1256 (1956).
(5) L. ONSAGER, *Phys. Rev.*, Vol. 37, p. 405 (1931); *ibid.*, Vol. 38, p. 2265 (1931).
(6) E. H. SONDHEIMER, *Proc. Roy. Soc. A*, Vol. 234, p. 391 (1956).
(7) D. ENSKOG, dissertation, University of Utrecht, 1917.
(8) D. DORN, *Zeits. f. Naturf.*, Vol. 12a, p. 739 (1957).
(9) M. KOHLER, *Ann. Physik*, Vol. 40, p. 165 (1941).
(10) F. G. MOLINER AND S. SIMONS, *Proc. Camb. Phil. Soc.*, Vol. 53, p. 848 (1957).
(11) G. E. TAUBER, *Can. J. Phys.*, Vol. 36, p. 1308 (1958).
(12) G. E. TAUBER, AND L. SOFFER, *J. Phys. Chem. Solids*, Vol. 8, 138 (1959).

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MEALS, R. N. AND LEWIS, F. M. Silicones. 1959.

NOYES, ARTHUR AMOS AND MULLIKEN, SAMUEL P. Laboratory Experiments on the Class Reactions and Identification of Organic Substances. Ed. 3. 1915.

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BERKELEY, EDMUND CALLIS. Symbolic Logic and Intelligent Machines. 1959.
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CARNAP, RUDOLF. Introduction to Symbolic Logic and its Applications. 1958.
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RODGERS, THOMAS M. Handbook of Practical Electroplating. 1959.

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ALLENDOERFER, CARL B. AND OAKLEY, CLETUS O. Fundamentals of Freshman Mathematics. 1959.
BATEMAN, HARRY. Partial Differential Equations of Mathematical Physics. 1959.
JAMES, GLENN AND JAMES, ROBERT C. Mathematics Dictionary. Multilingual Ed. 1959.
JEENEL, JOACHION. Programming for Digital Computers. 1959.
LANG, SERGE. Abelian Varieties. 1959.
MCCORMICK, EDWARD MACK. Digital Computer Primer. 1959.
RICE, HAROLD S. AND KNIGHT, RAYMOND M. Technical Calculus and Analysis. 1959.
TAYLOR, ANGUS ELLIS. Calculus, with Analytical Geometry. 1959.
ZYGMUND, ANTONI. Trigonometric Series. 1959.

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BEHRENS, CHARLES FREDERICK. Atomic Medicine. Ed. 3. 1959.

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Advances in Steel Technology in 1956. 1958.
BETTERIDGE, W. The Nimonic Alloys. 1959.

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DEFANT, ALBERT AND DEFANT, FRIEDRICH. Physikalische Dynamik der Atmosphäre. 1958.
MASSEY, HARRIE STEWART WILSON AND BOYD, R. L. F. The Upper Atmosphere. 1959.

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MISSILES AND ROCKETS

Rocket Encyclopedia. 1959.
Space Weapons; a Handbook of Military Astronautics. 1959.
ULANOFF, STANLEY. Illustrated Guide to U. S. Missiles and Rockets. 1959.

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U. S. ATOMIC ENERGY COMMISSION. Progress in Peaceful Uses of Atomic Energy; July–December, 1957. 1958.

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BELL, CARLOS G., JR. Liquid Scintillation Counting. 1958.
BOGOLIUBOV, N. N.; TOLMACHEV, V. V. AND SHIRKOV, D. V. A New Method in the Theory of Superconductivity. 1959.
HAYES, WALLACE D. AND PROBSTEIN, RONALD F. Hypersonic Flow Theory. 1959.
HOOKER, K. H. AND WEIMER, K., ED. Lexikon der Kern-und Reaktortechnik. Vol. 1 and 2. 1959.
HOTCHKISS, HOMER J. AND FAIRBANKS, FLOYD C. Principles of Experimental Physics. 1913.
MARSHAK, ROBERT E. Meson Physics. 1952.
SCOTT, WILLIAM TAUSSIG. The Physics of Electricity and Magnetism. 1959.
SHORTLEY, GEORGE HIRAM AND WILLIAMS, DUDLEY. Principles of College Physics. 1959.
UNIVERSITÉ DE GRENOBLE. The Many Body Problems. 1959.
VON HIPPEL, ARTHUR ROBERT. Molecular Science and Molecular Engineering. 1959.
WIGNER, EUGENE P. Group Theory and its Application to the Quantum Mechanics of Atomic Spectra. 1959.

RESEARCH

AMERICAN CHEMICAL SOCIETY. COMMITTEE ON PROFESSIONAL TRAINING. Directory of Graduate Research. 1957. 1958.

NATIONAL SCIENCE FOUNDATION. Scientific Research and Development in Colleges and Universities; Expenditures and Manpower, 1953–54. 1958.

SANITARY ENGINEERING

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MATTERS PERTAINING TO ELIMINATION OF BACKGROUND IN LOW ENERGY NUCLEAR PHYSICS MEASUREMENTS, WITH SPECIAL REFERENCE TO THE BARTOL ONR GENERATOR*†

BY

C. P. SWANN, V. K. RASMUSSEN AND H. O. ALBRECHT

ABSTRACT

A new low background target room has been constructed for the Bartol-ONR Van de Graaff generator. This room is constructed of low radioactivity content concrete and serves to shield the target area from the radiation arising from the soil, the beam analyzing magnet, and the small Bartol generator. Over the energy range from 400 kev to 3 Mev, the natural radioactive background is at least a factor of 5 lower than in the original target room. The effects of neutrons from the analyzing magnets and the small machine have been reduced to a negligible amount. The cost of the special concrete used in this structure was less than twice the cost of normal concrete.

INTRODUCTION

Certain problems involving the reduction of natural radioactive background and background from other accelerators have arisen in connection with the Bartol-ONR Van de Graaff generator. While these are hardly unique problems, we found that the published literature did not seem to contain much that was directly applicable to our situation.[1] The following is written, therefore, with the hope that a somewhat detailed description of our solution of these problems might be useful to others in the field of low-energy nuclear physics.

GENERAL NATURE OF THE PROBLEM AS APPLIED TO THE BARTOL-ONR ACCELERATOR

The Bartol-ONR 5.5 million volt Van de Graaff generator built at the Bartol Research Foundation of The Franklin Institute has now been in operation for seven years. During this period, there has been a continuing program to reduce the γ-ray and neutron background resulting from the accelerated particles striking parts of the vacuum plumbing other than the intended target. This program, initiated as part of the nuclear resonance fluorescence studies, has been successful to the extent that in a typical resonance fluorescence measurement the major part of the background was independent of whether or not the accelerator was running; that is, this background resulted from the natural radioactive

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† The authors wish to express their appreciation of the support and encouragement given to this project by the retiring Director of the Bartol Research Foundation, Dr. W. F. G. Swann, and of the advice and assistance by our colleague, Dr. F. R. Metzger.

[1] The authors are indebted to Dr. H. E. Gove for a helpful discussion of the solution of similar background problems at Chalk River.

contamination of the building and of the underlying soil. The various possibilities for absorbing out this radiation were considered, but the technical problems and cost seemed far too great.

The building housing the Bartol-ONR accelerator also houses the 1.7 million volt Bartol Van de Graaff, and the target areas for the two machines are separated only by a 2-ft. thick concrete block wall (see Fig. 1). The smaller machine is used as a neutron source, and when it is in operation experiments with the larger machine involving detection of either neutrons or γ-rays are interfered with severely. Induced radioactivities may also be troublesome for several hours after the smaller machine is turned off. Serious limitations have thus been imposed on the operating time of both machines. Means by which the shielding between the two target areas could be improved were considered, but even the best, which would have been very expensive, would not have reduced the neutron effect to an amount small compared with the natural room background.

The use of beams other than protons in the Bartol-ONR machine was also banned for two reasons. First, the vacuum system becomes contaminated with the gas used in the ion source, and it may take many hours of operation with hydrogen before the system is cleaned out. Second, the radioactivity induced by these other beams can be appreciable and may take days to decay to a low level.

A solution to all of the problems appeared to be the construction of a new target room for the large generator. This room had to be adequately shielded against the neutron effects of the small machine and the radioactivity of the ground and had to be constructed of low Z, low radioactivity content materials. The low Z was desirable to reduce the intensity of elastically scattered gamma rays which might add appreciably to the background of the resonance fluorescence studies.

Only proton beams should be used in this room while the old room could be used for other particles as well as for all charged particle spectrometer studies.

In order to determine the feasibility of meeting the requirements for this new room, several studies were conducted. First, the radioactivity within the soil around the main building was measured, and the amount of shielding to absorb out this radiation was determined. Second, the neutron intensities resulting from the small machine were measured at several points around the building, and the required additional shielding necessary to reduce this to a low level was also measured. Third, studies were undertaken to determine the type of construction of the new room which would give a low background for a reasonable cost.

ROOM LOCATION AND ABSORPTION

The radioactivity of the soil was measured by lowering an NaI detector into a 3-ft. deep hole in the ground. The presence of 2.62-Mev

and 1.46-Mev gamma rays from thorium and potassium, respectively, was quite apparent, and the background of this origin was somewhat greater than in the target toom. It was further determined that 2 in. of lead or 12 in. of normal concrete would reduce this background to a reasonable amount.

Figure 1 shows the arrangement of the main building and the points *A*, *B*, and *C* indicate locations at which the neutron intensities resulting from the small generator were measured. It also shows the location of the new target room. The counting rate dropped from *A* to *B* to *C*, as would be expected because of the better shielding, and the rate at *C*

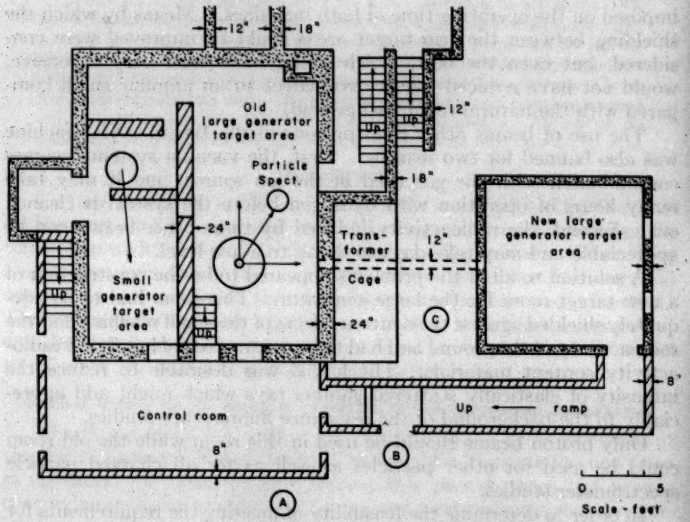

FIG. 1. Floor plan of the old and new target areas.

appeared to be low enough to enable one foot or so of concrete to reduce the effect of the remaining neutrons to an acceptable amount. In designing a new target room, and in order to maintain the same floor level as in the main building and a reasonable ceiling height of 9 ft., about 3 ft. of the room plus the ceiling thickness had to be above ground. It was realized that because of uncertainties in the measurements, 1-ft. concrete walls and ceiling might not be sufficient, but it was expected that enough dirt could be piled around and on top of the protruding structure to take care of this possibility. Background measurements upon completion proved this to be unnecessary.

CONCRETE STUDIES

In the appendix is given a complete report on the concrete studies. In brief, sources of low radioactivity content sand and gravel were located and concrete blocks were prepared from these ingredients. From these a cell of about 1 cubic meter volume was prepared and the radiation within this cell measured and compared with that in the old target room and that in a lead cell.[2] This block cell, over the energy range from 440 kev to 3 Mev, gave a reduction in background by a factor of 3 to 4 over the corresponding background in the old target room. This factor was not corrected for the natural radioacivity within the NaI crystal and photomultiplier and for leakage through the cracks of the cell. However, even a factor of 3 in reduction seemed desirable considering the other advantages of the new room. Since the cost of the special concrete as compared to the cost of normal concrete was not excessive, plans were drawn up for the new room and estimates of cost obtained. These figures came within the allowable amount. Therefore, the project was approved. Figure 1 shows the general features of this room.

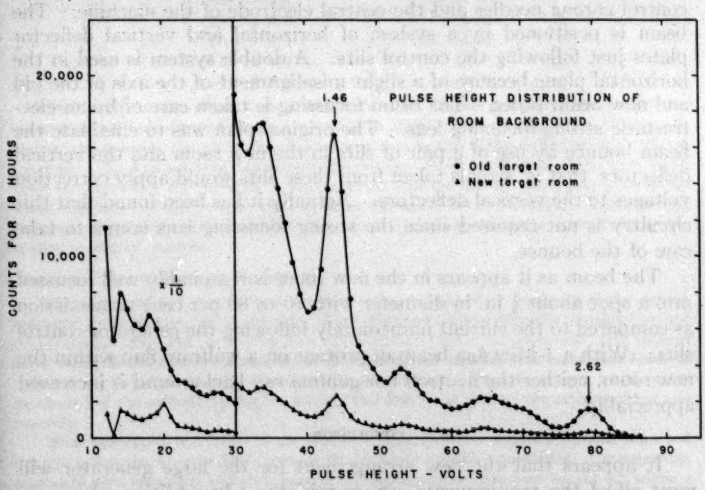

FIG. 2. Pulse-height distribution of the γ-ray background in the old and new target rooms. The indicated energies are those for the photo peaks of K^{40} (1.46 Mev) and thorium (2.62 Mev).

[2] It will be observed that in view of the law of inverse squares, and of the fact that the area of the walls of the cell are proportional to the square of the linear dimensions, observations of background taken in a cell will, to a first approximation, be independent of its size for an assigned wall thickness.

Upon completion, the natural background within the new room was measured using a 100 channel analyzer. The results are shown in Fig. 2 along with the spectrum obtained in the old target room. It may be seen that the background in the new room is lower than that in the old room by a factor of at least 5, and without considering the natural radioactivity within the crystal and photomultiplier. Preliminary tests also indicated that the effect of the neutrons produced by the small generator is very low.

BEAM EXTENSION SYSTEM AND OPERATION

While the new room was under construction, the system required to bring the charged particle beam into this room was being developed. Since the targets to be used in the new room would be about 35 ft. from the exit control slits of the analyzing magnet, provisions had to be made for positioning and focussing of the beam. Also, it was evident that means might have to be developed to take out the bounce of the beam between the control slits. This bouncing is the result of the relatively slow regulation caused by the large distance between the control corona needles and the central electrode of the machine. The beam is positioned by a system of horizontal and vertical deflector plates just following the control slits. A double system is used in the horizontal plane because of a slight misalignment of the axis of the old and new beam tubes. The beam focussing is taken care of by an electrostatic strong focussing lens. The original plan was to eliminate the beam bounce by use of a pair of slits in the new room and the vertical deflectors, that is, signals taken from these slits would apply correction voltages to the vertical deflectors. Actually it has been found that this circuitry is not required since the strong focussing lens seems to take care of the bounce.

The beam as it appears in the new room is reasonably well focussed into a spot about $\frac{1}{8}$ in. in diameter with 70 to 80 per cent transmission as compared to the current immediately following the generator control slits. With a 3-Mev 4μa beam of protons on a wolfram flap within the new room, neither the neutron nor gamma ray background is increased appreciably.

CONCLUSION

It appears that the new arrangement for the large generator will meet all of the requirements set up initially. In addition, there are other advantages which have become apparent. Actually there is considerably more space in this new room and less scattering material, that is, the beam analyzer magnet and the charged particle spectrometer are absent. Also, the long beam path provides for convenient methods of pulsing the beam.

APPENDIX

The information here given may be of service to others who are faced with problems analogous to that here involved. To recapitulate, the elimination of background envisaged the construction of a new target room for the large generator, a room isolated by distance and shielding from the effects of the two machines located in the main building. This room had to be partially underground, and since the soil is quite radioactive, the walls and floor of the room had to be sufficiently thick to absorb out the soil radiation and yet be low in activity themselves.

Low Background Concrete

The following types of construction have been considered:

1. Poured concrete
2. Timber with backfill of sand
3. Marble lining of (1)
4. Silica brick (asphaltic concrete)

The second method seem impractical and was not considered further. The third method seemed feasible if the first should not turn out too well; in any case, marble could be added later.

Cost of Silica Brick

Because silica brick might well be somewhat lower in radioactivity than a special concrete, the following cost figures were obtained from General Refractories:

Size in inches	$ per cu. ft.
15 × 6 × 3	4.06
15 × 9 × 3	4.30
12 × 6 × 3	3.50
12 × 9 × 3	3.78

The density of silica brick is about 103 lbs/cu. ft. Normal poured concrete would cost about $0.54 per cubic ft. and the density would be about 150 lbs. per cubic ft. Taking into account the densities, the cost of the silica bricks would be 8 to 10 times the cost of the poured concrete. This does not take into account the relative cost of installation which again would favor the concrete. Furthermore, it was expected that the cost of the special concrete we would require would not be more than 1½ times the cost of normal concrete. Consequently, the use of silica bricks was not considered further. The remainder of the report concerns the studies on poured concrete.

Materials of Low Radioactivity for Concrete

Potassium and the thorium and uranium series provide all or nearly all of the natural radioactivity to be expected in concrete. Gray Portland cement normally contains ½–¾ per cent potassium oxide and can only by good luck be chosen from a low potassium batch without extra cost. White cement, according to the one sample treated, runs somewhat lower in radioactivity but at about twice the cost of the gray cement. Thus the cement, which makes up about ⅛ of the concrete by weight, sets a rather firm lower limit on the activity of low cost concrete.

Sand and gravel or crushed stone offer two possibilities for selection because silica and calcium carbonate do occur fairly pure in commercially available quantities. Consequently, sources of pure silica sand and gravel and of crushed limestone have been sought out within a reasonable distance. The local gravel and stone is, for the most part, river gravel or "granite." The former contains pebbles of all sorts, many containing feldspar which in turn contains potassium. "Granite" contains much feldspar and often traces of uranium and thorium minerals.

Sand which is nearly pure silica is comparatively common since silica is resistant to leaching and attrition. Gravel, even if white and essentially silica, will contain more inclusions of radioactive grains.

Samples were obtained from local areas which supplied all of the local concrete mixing plants and from more distant sources towards the glass-producing area of New Jersey.

Testing of Materials

The various materials were tested by filling two tin pans with the samples and placing these pans in a lead enclosure in constant relationship to a 2 × 1¾-in. NaI detector. The two pans held about 11 lb. of cement, 17 lb. of sand, or 16 lb. of gravel. The justification for using comparatively small spot samples of concrete ingredients are:

1. The materials have been fairly well mixed, both by local geological processes, and screening and washing operations.

2. It would be practically impossible to arrange for the use of exactly the material from which a more elaborate sample was taken.

3. A few duplicating samples taken at different times during this work gave similar results.

Of course it is always possible that the sample may include an exceptional lump of uranium or thorium material in a single stone. The use of duplicating samples however, eliminated such a chance occurrence.

The pulses from the cathode follower of the photomultiplier were fed into a modified Atomic Instruments Model 218 linear amplifier. This amplifier has two discriminators. One discriminator was set for pulses greater than 440 kev and the other for pulses greater than 880 kev. The high level output was fed into a single channel pulse height analyzer which was set for pulses greater than 1540 kev. With these pulses fed into three scalers, the number of pulses in the intervals of 440 to 880 kev and 880 to 1540 kev were determined. The gain was reduced by a factor of two to give the pulses in the intervals from 880 to 1760 kev and 1760 to 3080 kev.

All crucial samples were run sufficiently often in juxtaposition with blank runs to give assurance of the meaningfulness of the results for present purposes. The errors in sampling and procuring of these materials would not justify more elaborate precautions or the expenditure of more time.

The results for a number of samples are given in Table I. Several samples are included

TABLE I.—*Radioactivity in Various Materials.* (*Counts with Sample in Place Minus Counts without Sample.*)

Sample and No.	Counts per 10-Min. Interval 0.44 to 0.88 Mev	0.88 to 1.54 Mev	0.88 to 1.76 Mev	1.76 to 3.08 Mev
16 Allentown cement	133	63	62	0
4 White cement	74	23	25	8
1 Wenonah sand	108	98	97	−3
3 Pine Valley sand	333	29	34	8
12 Millville bar sand	13	2	24	−3
13 Millville coarse sand	55	13	13	−3
2A Tallytown 1-in. gravel	63	87	77	19
3 Pine Valley gravel	30	45	21	−2
6 Valley Forge limestone	102	10	31	−3
7 Cedar Hollow limestone	113	11	—	—
9 Wenonah ¾-in. gravel	309	61	73	8
11 Millville ¾-in. gravel	63	−8	−9	1
14 Millville ¾-in. gravel	21	−8	−1	8
18 Chickies quartzite	248	118	146	14
21 Glen Mills 1-in. stone	277	152	176	8
15 Ironstone	475	181	206	72
17 Pegmatite	1067	1029	1044	46
20 1.4 lb. potassium salts	742	1130	1062	16
22 Soil south of generator	246	109	102	10

at the end for general information. It appears from these last results that the precipitated iron from the leaching solutions acting on the coastal plain sediments may concentrate considerable radioactivity, and that the lightest possible colors of sand and gravel are therefore desirable. The desirability of eliminating pegmatite (light colored veins) from crushed stone is also indicated.

The best sand and gravel appear to be those from the Millville Sand and Gravel Company at Port Elizabeth, New Jersey, whereas the best cement is the white variety. For the concrete test blocks studied in the next section, the Millville bar sand and ¾-in. gravel and the gray cement were used, the latter for cost reasons as compared to the ultimate effect.

Background in Concrete Enclosure

Twenty concrete blocks 8 × 8 × 12 in. in size were made using a 1:2:3 mix by volume. Six of these test blocks were laid on a platform of 3 in. of lead. Eight more blocks on end were placed on this square to form an 8 × 8 × 12-in. enclosure. Six more blocks, arranged like the bottom, constituted the cover.

The results of placing the scintillation counter in this enclosure as compared to a lead enclosure and the open room are given in Table II. The open room corresponds to the activity

TABLE II.—*Background Measurements.**

	Counts per 10-Min. Interval		
	0.44–0.88 Mev	0.88–1.54 Mev	1.76–3.08 Mev
Counter on floor of main building	9855	5346	994
Counter in lead shield	509	262	69
Counter in 8 × 8 × 12-in. concrete enclosure	3146	1199	267

* No correction has been made for activity within the crystal.

of the concrete of the main building plus the amount of radiation from the soil coming through the floor and walls. The results were well reproducible on successive 10-min. runs, the errors on the larger figure being about ±2.5 per cent and those on the lower being about 10 per cent. Two effects must be considered in determining the real effect of the concrete enclosure. First, cracks exist between the blocks and around the leads to the photomultiplier; in the case of the lead enclosure there was better overlapping of the lead blocks. Second, undoubtedly much of the counting rate within the lead enclosure is from the NaI crystal itself. This effect, whatever it may be, should be subtracted from all of the results, and the result of such correction is to lower the estimated background to an estimate which, percentage wise, is greater for small background than for large ones.

Material Requirements and Cost

Before making a final decision on the construction of this new room, some idea of the cost of the special concrete must be made. It had already been decided that the walls, floor and ceiling must be one foot thick and that the inside dimensions should be 20 ft. square by 9 ft. high. The amount of concrete required was then:

	Vol., cu. ft.
Floor and roof 22 × 22 ft.	968
Four walls 9 × 21 ft.	756
	1724 cu. ft.

With a density of 149 lb./cu. ft. this gives a total weight of about 129 tons of concrete. The mix decided upon—one part cement, two parts sand, and three parts gravel—has a compressive strength rating of 2800 psi. and costs between $14 and $15 per cubic yard. This would give as the concrete cost for normal material of about $1000.

The actual amounts of the individual components are about 17 tons of cement, 43 tons of sand, and 76 tons of gravel. The following costs were quoted by the Millville Sand and Gravel Company at the Port Elizabeth, New Jersey pit.

	$/ton
Pure bar sand	0.60
¾-in. gravel	1.80
Freight to Philadelphia	3.20

Our presumed supplier of this concrete gets his sand from Wenonah, New Jersey, and his gravel from Glen Mills, Delaware County, Pennsylvania. Therefore, in estimating the greater cost of materials we may say that the freight or trucking cost on the sand would be about doubled while that for the gravel would be about five times. Therefore, the extra charge per cubic yard of concrete would be about $4.00 without allowing for special runs of the mixing, and storage of the special sand and stone. This latter is important since it may take as long as six weeks to make all the pours. It is not expected however that the cost will more than double, that is, not more than $30 per cubic yard.

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CONTRIBUTIONS BY STAFF MEMBERS

FIRST SUPPLEMENT[1]

Air Pollution

200. "Auto Exhaust: Composition and Photolysis Products," by E. R. Stephens, P. L. Hanst, R. C. Doerr and W. E. Scott, *J. Air Poll. Cont. Assn.*, **8**, 333 (1959).

Air Traffic Control

60. "Some Channel Allocation Problems in Air-Ground Voice Communications," by W. W. Felton, *Trans. IRE*, **ANE-1**, 41 (1954).

194. "Simulation Studies of Air Traffic Control Problems in High-Density Areas," by T. K. Vickers* and R. S. Miller, in "Proc. 5th Annual IRE East Coast Conf., Baltimore, 1958," p. 173.

Chemistry

128. "Study of Ignition by Radiation and Natural Convection in Confined Atmospheres," by H. G. Bickford and G. P. Wachtell, in "Proc. 2nd Symposium on Solid Propellant Ignition, Oct. 1956," p. 89.

189. "First Limit Induction Time Studies of CS_2-O_2 Explosions," by F. R. Taylor and A. L. Myerson, in "Proc. 7th International Symposium on Combustion," Butterworths, 1958, p. 72.

199. "A New Reaction of Organic Peroxyacids," by M. M. Labes, *J. Organ. Chem.*, **24**, 295 (1959).

Colloids

204. "Surface Energy and Adhesion Properties in Asphalt-Aggregate Systems," by E. Thelen, Highway Research Board Bulletin 192, Jan.6–10. 1958.

Computers

228. "A Rapid Statistical Data Processing System for Radio Propagation Research," by S. Charp, in "Proc. Symposium on Statistical Methods in Radio Wave Propagation," U. of Calif., 1959.

Engineering Psychology

197. "A Review and Summary of Tracking Research Applied to the Description of Human Dynamic Response," by D. T. McRuer and E. S. Krendel, in "IRE WESCON Convention Record, Part 4, 1958," p. 254.

207. "A Survey of Suggested Mathematical Methods for the Study of Human Pilot's Responses," by E. S. Krendel, in "Report of 2nd Piloted Aircraft Flight Control System Symposium, June 1952," BuAer Report AE-61–5-III.

208. "Human Dynamics and System Synthesis," by E. S. Krendel and D. T. McRuer, in "Proc. of National Syms posium on Human Factors in System-Engineering, HFSA, IRE, Phila., Pa. Dec., 1957."

209. "Human Transfer Functions," by E. S. Krendel, in "Proceedings, 2nd Annual Human Factors Discussion Group, 11th Annual International Air Safety Seminar, Nov. 1958," Flight Safety Foundation, New York, p. 17.

224. "The Human Operator as a Servo System Element," by D. T. McRuer and E. S. Krendel, JOUR. FRANKLIN INST., **267**, 381 and 511 (1959).

Friction-Lubrication

11. "What About Friction?: I, Classical Laws; II, Friction as the Result of Molecular Forces; III, Friction Saw, Electrical Theory, Lubrication," by F. Palmer, *Am. J. Phys.*, **17**, 181, 327, 336 (1949).

36. "Mixed Friction Conditions in Lubrication," by D. D. Fuller, *Lubrication Eng.*, Sept–Oct. 1954.

[1] This list supplements the list of contributions published by staff members as printed in this JOURNAL for January, 1959. Asterisks indicate co-authors from other organizations. The numbered contributions (listed by subject) are available as reprints, on request to the Laboratories; these include a few, listed previously, for which complete references have become available. Reprints Nos. 43, 98, 99, 115 and 117 are no longer available. The unnumbered contributions (listed separately beginning on page 237) are not currently available. Please indicate by number the reprints you wish to receive.

155. "A Study of the Effect of Wear Particles and Adhesive Wear at High Contact Pressures," by E. B. Sciulli and G. M. Robinson, *Trans. ASLE*, **1**, 312 (1958).
166. "Determination of Optimum Proportions for Hydrostatic Bearings," by A. M. Loeb and H. C. Rippel, *Trans. ASLE*, **1**, 241 (1958).
211. "The Influence of the Molecular Mean Free Path on the Performance of Hydrodynamic Gas Lubricated Bearings," by A. Burgdorfer, *Trans. ASME, J. Basic Eng.*, **1**, 94 (1959).
215. "Axial, Relative Motion of a Circular Step Bearing," by L. Licht, *Trans. ASME, J. Basic Eng.*, **1**, 109 (1959).

General Physics

203. "Technical Data on Border-Punched Cards," by G. Cohn, *Am. Documentation*, **X**, 116 (1959).
206. "The Theory of Low-Temperature Bolometer Detectors Applied to the Measurement of Low-Level RF Power," by D. L. Birx and N. Fuschillo, *Trans. IRE Instrumentation*, **I-7**, 310 (1958).
213. "Early Detection of Fatigue in Metal Alloys by Ultrasonics," by K. Sittel, *Nondestructive Testing*, **XVII**, 165 (1959).
216. "On the Reaction of Fluids and Fluid Jets," by R. Ekesrgian, JOUR. FRANKLIN INST., **237**, 385 (1944).

Magnetism

139. "Magnetic Properties of Some Iron and Iron Oxide Micropowders," by R. B. Campbell, H. Amar, A. E. Berkowitz and P. J. Flanders, AIEE Spec. Pub. T-91, 1956.
222. "Magnetic Properties of Some Ferrite Micropowders," by A. E. Berkowitz and W. J. Schuele, *J. Appl. Phys.*, **30**, 134S (1959).

Mechanical Engineering

230. "Vibration of Beam Mass Systems with Time-Dependent Boundary Conditions," by T. C. Yen and S. Kao,* ASME Paper No. 59-APM-22.

Metallurgy

202. "Cellular Substructure in Zn Crystals Grown from the Melt," by V. Damiano and M. Herman, *Trans. AIME*, **215**, 136 (1959).
212. "Etch Patterns in Zinc," by V. Damiano and M. Herman, JOUR. FRANKLIN INST., **267**, 303 (1959).
221. "Precipitation in a Beta-Brass-Fe Alloy," by A. E. Berkowitz and P. J. Flanders, *J. Appl. Phys.*, **30**, 111S (1959).

Metals Physics

110. "Self-Diffusion in Single and Polycrystals of Zinc at Low Temperatures," by F. E. Jaumot, Jr. and R. L. Smith, *J. Metals*, **8**, 164 (1956).
193. "Preparation of Fine Particles from Bimetal Oxalates," by W. J. Schuele, *J. Phys. Chem.*, **63**, 83 (1959).
220. "Properties of Multidomain Particles," by H. Amar, *J. Appl. Phys.*, **30**, 139S (1959).

Plastic Deformation of Metals†

201. "A Study of Dislocations in Thin Aluminum Foils Elongated in the Electron Microscope," by H. G. F. Wilsdorf, in "Symposium on Advances in Electron Metallography," Spec. Tech. Publ. No. 245, ASTM, 1958, p. 43.
214. "A Study of Slip Lines in Alpha-Brass as Revealed by the Electron Microscope," by J. T. Fourie* and H. G. F. Wilsdorf, *Acta Met.*, **7**, 339 (1959).
218. "Dislocation Patterns in Deformed Alpha-Brass," by H. G. F. Wilsdorf, to be published in "Proc. Symposium on Internal Stresses and Fatigue in Metals," Elsevier Publ. Co., 1959, in press.
227. "Experimental Techniques for the Plastic Deformation of Metal Foils in the Electron Microscope," by H. G. F. Wilsdorf, L. Cinquina, and C. J. Varker to be published in "Proc. International Congress of Electron Microscopy, 1958."
229. "The Effect of Plastic Deformation on the Resistivity of Copper-Palladium Alloys," by E. Klokholm and B. Hyatt, to be published in *Trans. AIME*.

Rheology

205. "Use of Rheological and Other Data in Asphalt Engineering Problems," by J. G. Brodnyan, Highway Research Board Bulletin 192, Jan. 6–10, 1958.

† NOTE: Reprints Nos. 111, 150, 177, previously listed, should be classified under Plastic Deformation of Metals.

V.2
195

217. "The Rheology of Various Solutions of Cellulose Derivatives," by J. G. Brodnyan, F. H. Gaskins, W. Philippoff and E. G. Lendrat,* to be published in *Trans. Soc. Rheology*, II, in press.

219. "The Capillary Experiment in Rheology," by W. Philippoff and F. H. Gaskins, to be published in *Trans. Soc. Rheology*, II, in press.

223. "The Experimental Check of Theories of the Viscosities of Solutions," by W. Philippoff and F. H. Gaskins, *J. Phys. Chem.*, **63**, 985 (1959).

225. "Instrumentation for Rheological Investigation of Viscoelastic Materials," F. H. Gaskins and W. Philippoff, *Ind. Eng. Chem.*, **51**, 871 (1959).

226. "Stress-Optical Analysis of Fluids," by W. Philippoff, *Ind. Eng. Chem.*, **51**, 883 (1959).

Semiconductors

196. "Transport Properties of the Pseudo-Binary Alloy System $Bi_2Te_{3-y}Se_y$," by N. Fuschillo, J. N. Bierly and F. J. Donahoe, *J. Phys. Chem. Solids*, **8**, 430 (1959).

198. "Transport Phenomena in Semiconductors for Warped Energy Surfaces," by G. E. Tauber and L. Soffer,* *J. Phys. Chem. Solids*, **8**, 138 (1959).

210. "A Generalized Variational Principle for Transport Phenomena," by G. E. Tauber, *Canadian Journal of Physics*, **36**, 1308 (1958).

Thermoelectricity

231. "Theory of the A. F. Joffe Method for Rapid Measurement of the Thermal Conductivity of Solids," by W. F. G. Swann, JOUR. FRANKLIN INST., **267**, 363 (1959).

CONTRIBUTIONS FOR WHICH REPRINTS ARE NOT CURRENTLY AVAILABLE

"Convertible Aircraft," by R. S. Barnaby, S. M. Berkowitz and W. H. Colcord, *Aero. Eng. Rev.*, **8**, 4 (1949).

"The Micelle and Swollen Micelle on Soap Micelles," by W. Philippoff, *Faraday Soc. Disc.*, No. 11, 96 (1951).

"Reactions Involving Ozone Nitrogen Dioxide, and Organic Compounds at Low Concentrations in Air," by P. L. Hanst, E. R. Stephens and W. E. Scott, *J. Air Poll. Cont. Assn.*, **5**, 219 (1956).

"Recent Developments in the Study of the Organic Chemistry of the Atmosphere," by E. R. Stephens, W. E. Scott, P. L. Hanst and R. C. Doerr, API Reprint, 1956.

"Journal-Bearing Operation at Superlaminar Speeds," by M. I. Smith* and D. D. Fuller, *Trans. ASME*, **78**, 469 (1956).

Theory and Practice of Lubrication for Engineers, by Dudley D. Fuller, John Wiley & Sons, Inc., 1956.

"Film-Pressure Distribution in Grease-Lubricated Journal Bearings," by G. Cohn and J. W. Oren,* *Trans. ASME*, **71**, 555 (1949).

"The Design of Lightweight Trains," by R. Eksergian, *Trans. ASME*, **56**, 667 (1934).

"The Dynamical Analysis of Machines," by R. Eksergian, International Congress of Applied Mechanics, Stockholm, 1930.

"The Fluid Torque Converter and Coupling," by R. Eksergian, JOUR. FRANKLIN INST., **235**, 441 (1943).

"General Dynamical Considerations Applied to Piezo-Electric Oscillations of a Crystal in an Electrical Circuit," by W. F. G. Swann, JOUR. FRANKLIN INST., **242**, 167 (1946).

"Certain Matters Concerning Scaling in the Magnetron with Special Reference to the Relative Efficiency of Magnetrons of Different Sizes," by W. F. G. Swann, JOUR. FRANKLIN INST., **246**, 149 (1948).

"Fluctuations in Proportional Counters," by W. F. G. Swann, JOUR. FRANKLIN INST., **249**, 133 (1950).

"Electrets," by W. F. G. Swann, *Phys. Rev.*, **78**, 811 (1950).

"On Certain Matters Pertaining to Electrets," by W. F. G. Swann, JOUR. FRANKLIN INST., **250**, 219 (1950).

"Engineering and Pure Science," by W. F. G. Swann, *Physics Today*, **4**, 9 (1951).

"The Engineer and the Scientist," by W. F. G. Swann, *Jour. Engineering Education*, **43**, 241 (1952).

"Fundamentals in the Behavior of Electrets," by W. F. G. Swann, JOUR. FRANKLIN INST., **255**, 513 (1953).

"The Case of Charged Density Distribution Versus Semi-Permanent Polarization as a

Basis for Electret Behavior," by W. F. G. Swann, JOUR. FRANKLIN INST., **256**, 167 (1953).

"Engineering in our Epoch," by W. F. G. Swann, University of Toronto Press, 7th Wallberg Lecture, 1954.

"Stress and Strain," by W. F. G. Swann, *J. Appl. Phys.*, **25**, 1108 (1954).

"Creative Intuition in Engineering Science," by W. F. G. Swann, Am. Management Assn., Report No. 3, Planning Ahead for Profit, 1957, p. 156.

"The Formation of Mechanical Twins," by N. Thompson* and M. Hingley, *Acta Met.*, **3**, 289 (1955).

"On the Representation and Extrapolation of Creep Data of Metals and Technical Alloys," by P. Feltham* and J. D. Meakin, *Rheologica Acta*, **2**, 176 (1958).

"Creep in Copper," by P. Feltham* and J. D. Meakin, *Acta Met.*, in press.

"Quenching Stresses and Strains in Ionic Crystals," by B. H. Kear and P. L. Pratt,* *Acta Met.*, **6**, 457 (1958).

"Quench-hardening in Sodium Chloride Crystals," by B. H. Kear and P. L. Pratt,* *Phil. Mag.*, **4**, 56 (1959).

"Some Dislocation Interactions in Simple Ionic Crystals," by B. H. Kear, A. Taylor* and P. L. Pratt,* *Phil. Mag.*, **4**, 665 (1959).

"Work-hardening in Sodium Chloride Crystals," by B. H. Kear and P. L. Pratt,* in press.

"The Half Life of I^{131} Labeled Normal Human Gamma Globulin in Patients with Hepatic Cirrhosis," by J. N. Bierly, P. Havens,* W. Dickensheets* and T. Eberhard,* J. *Immunology*, **73**, 256 (1950).

"The Mechanism of Exchange of Hydrogen between Nitrogen and Oxygen Atoms," by C. G. Swain,* J. T. McKnight,* M. M. Labes, and V. P. Kreiter,* *J. Am. Chem. Soc.*, **76**, 4243 (1954).

"The Mechanism of Exchange of Hydrogen between Ammonium and Hydroxyl Groups. I," by C. G. Swain* and M. M. Labes, *J. Am. Chem. Soc.*, **79**, 1084 (1957).

"Molecular Compounds. V. Solvent Effects on Complexing of 1,3,5-Trinitrobenzene with Aniline and with Naphthalene," by S. D. Ross* and M. M. Labes, *J. Am .Chem. Soc.*, **77**, 4916 (1955).

"Molecular Compounds. VI. A Reinvestigation of the Picryl Chloride-Hexamethylbenzene Complex; The Effect of Triethylamine on the Equilibrium Constant for Complex Formation," by S. D. Ross,* M. M. Labes and M. Schwarz,* *J. Am. Chem. Soc.*, **78**, 343 (1956).

"Molecular Compounds, VII. Interactions between 1,3,5-Trinitrobenzene and Pyridine-N-Oxides in Chloroform Solution," by S. D. Ross,* D. J. Kelley* and M. M. Labes, *J. Am. Chem. Soc.*, **78**, 3625 (1956).

"Molecular Compounds. VIII. Evidence for the Existence of 2:1 and 1:2-Complexes in Chloroform Solutions of 1,3,5-Trinitrobenzene and N,N-Dimethylaniline," by S. D. Ross* and M. M. Labes, *J. Am. Chem. Soc.*, **79**, 76 (1957).

"The Solvolyses of Some Tertiary Halides in Dimethylformamide and N-Methylpropionamide," by S. D. Ross* and M. M. Labes, *J. Am. Chem. Soc.*, **79**, 4155 (1957).

"The Dimethylamine-1,3,5-Trinitrobenzene Complex in Dioxane," by M. M. Labes and S. D. Ross,* *J. Org. Chem.*, **21**, 1049 (1956).

BOOK REVIEWS

FRAMES AND ARCHES, by Valerian Leontovich, with foreword by S. P. Timoshenko. 472 pages, diagrams, 6 × 9 in. New York, McGraw-Hill Book Co., Inc., 1959. Price, $20.00.

Even though the exact equations for the elastic analysis of arches was developed (by Weyrauch) as early as 1879, their complexity precluded their extensive applications to actual design problems, which were often solved by more or less elaborate graphical methods. This state of affairs was summarized in the transactions of the American Society of Civil Engineers in 1898:

"This modern and most exact method, however, is not free from criticism. While the fundamental principles of the theory are almost axiomatic, their final application to the solution of stresses is extremely complicated, so much so that few engineers can be credited with the patience and earnest endurance to master either the method or the solution of a problem to which it is applied. . . ."

Even today, the level of "patience and endurance" required for carrying out the analysis of indeterminate structures by such methods as slope deflection, virtual work and moment distribution is still considerable. Every practicing design engineer will therefore welcome the work of the present author, who can be credited with a major step towards the reduction of the work of computation, as well as the chances of error, in the analysis of frames and arches.

Condensed, yet clear solutions for shears and moments are given for twenty principal types of indeterminate structures. Over 400 loading conditions are dealt with, including concentrated and partial distributed loads, sidesway loads, and loads due to displacement of supports and changes in temperature. The simple algebraic operations developed by the author lead to determination of the redundants, from which the moments and forces for any section of the structure may be quickly calculated.

The book is divided into two main parts: the first comprising 12 sections, treats frames and arches with members of constant cross section, with both hinged and fixed supports, including portal frames, trapezoidal frames, gable frames, parabolic arches, and parabolic frames.

The second part presents what may be considered the first convenient and practical method of analysis for frames and arches of variable cross section. Sections 13 to 19 deal with frames with straight members. An extension of the method of elastic centers (called method of elastic parameters), developed by the author, not only allows treatment of members with one or two haunches, straight or parabolic, but enables the designer to analyze members not falling into these classifications as well. Portal, trapezoidal and gable frames with hinged and fixed supports are dealt with. Eight tables giving elastic parameters and load constants for the standard members are given in the Appendix.

Sections 20 to 24 treat frames with curved members, including parabolic arches and frames, with hinged and fixed supports, which may be solved with the help of 12 additional tables in the Appendix. These allow reduction to a standard form of any symmetrical arch, on the basis of the ratios of rise to span, and thickness at crown and springing. Factors for a great variety of loading conditions are tabulated, including concentrated, distributed and lateral loads, and factors for the effect of axial forces are also given for the three main types of arches considered by the author: arches of constant cross section; "prime" arches, in which the cross section varies at a more or less constant rate from the crown to abutments; and "quadratic" arches in which the cross section increases rapidly near the abutments.

These details can only give a superficial idea of the comprehensiveness of the work. Ample cross references are given, and a variety of examples are worked out in full to show the application of the author's method. The fig-

ures and notations are clear and unambiguous, and the attractive type does justice to the subject matter. An appealing feature is the inclusion, before each chapter, of photographs of large and unusual structures, either completed, or in the process of construction, or, in some cases, both. The concise captions draw attention not only to their salient engineering details, but also to their aesthetic aspects, thus showing again that something of an artist is hidden in every outstanding structural engineer.

K. L. CAPPEL
The Franklin Institute Laboratories

ČERENKOV RADIATION, by J. V. Jelley. 304 pages, diagrams, plates, 5¼ × 8¼ in. New York, Pergamon Press, 1958. Price, $10.00.

The appearance of this excellent book, the only monograph devoted exclusively to the subject of Čerenkov radiation, is indeed timely, since it coincides with the awarding of the 1958 Nobel Prize in Physics for the discovery of this phenomenon 24 years earlier.

The effect first manifested itself as a feeble bluish-white light emitted by transparent substances under the influence of the radiations from radioactive substances. When an electrically-charged particle, such as an electron, passes through a medium with a velocity exceeding the velocity of light in that medium, a kind of optical shock-wave is produced. This is quite analogous to the shock-wave created in air by a missile travelling at supersonic velocity, or the bow-wave from a ship moving faster than the velocity of the surface waves. The light output can be measured by a photomultiplier tube, and information about the nature and properties of the individual particle producing the pulse can thus be deduced.

The Čerenkov detector has recently become an important tool in the hands of the cosmic ray and nuclear physicist. In 1951, Dr. Jelley succeeded in detecting single fast charged particles for the first time by this method, and since then he has made numerous significant contributions to the development of the Čerenkov counter. An eminent authority on the subject, he is exceedingly well qualified to discuss Čerenkov radiation and its applications.

This comprehensive book will be a welcome addition to the reference library of the experimentalist who is interested in exploiting the possibilities of this new device. It contains a compendium of useful technical information, including some, such as tables summarizing characteristics of probably all commercially-produced photomultiplier tubes, which will require frequent revision. Relevant curves and data from standard reference sources with which workers in the field are thoroughly familiar are collected in the Appendix, and there is an extensive and apparently complete bibliography, including the considerable work carried out in the USSR.

From the point of view of fundamental physics, the treatment is essentially historical. Chapter 1 is a general introduction, followed by a presentation of theoretical considerations in Chapters 2 and 3. After a survey of later experimental work in Chapter 4, there is an abrupt switch to a discussion of the technology of Čerenkov counters in the next four chapters. These deal with photomultipliers, radiators, design factors involved in the most efficient transfer of light from the radiator to the multiplier, and descriptions of some practical counters. Although the logical justification for next inserting a chapter on Čerenkov radiation in the atmosphere is not apparent, this review article is of interest in its own right to at least one segment of the readers of this book, cosmic ray investigators. The text then again reverts to counters, discussing the little-used potentialities of the gas Čerenkov counter which should have some application in experiments conducted with the very large accelerating machines at present under construction. The final chapter is a potpourri of miscellaneous topics, some controversial, others highly speculative. Needless to state the author is competent to distinguish fact from fancy, and indeed does so.

For whatever it is worth, the reviewer would like to state that he has already derived much benefit from the use of Dr. Jelley's book in his own work.

MARTIN A. POMERANTZ
Bartol Research Foundation

SOUNDING ROCKETS, by Homer E. Newell, Jr. 334 pages, illustrations, 6 × 9 in. New York, McGraw-Hill Book Co., Inc., 1959. Price, $12.50.

Man lives at the bottom of a turbulent ocean of air. Everything coming through this atmosphere is attenuated, scattered or altered.

Beyond the atmosphere we find the full range of the electromagnetic spectrum. Thus, on the surface of the Earth most of these radiations have been filtered out. We are told that if the entire electromagnetic spectrum was considered a keyboard with 60 keys, one single key represents that which we can see with our eyes and but a few more represent what can be detected with instrumentation from the Earth's surface. To obtain a definitive picture of what energies are present in space and their flux, it is necessary to ascend beyond the atmosphere.

Until a few years ago this was impossible. The balloon was the only vehicle which could ascend to great heights to collect significant information. With the advent of the rocket this has been changed.

Today scientists all over the world have been probing radiations beyond the atmosphere. With the fantastic improvements in instruments we are able to parlay the short periods of time during which the sounding rocket spent above most of the atmosphere to draw a picture of what takes place in this region.

One of the true pioneers in this work is the senior editor of this new book. Dr. Newell began his work in this field when the U. S. Army looked for something to put into the payload sections of the V-2's which had been brought over from Germany. His work as head of the Rocket Sonde Branch of NRL's rocket upper-air program gave him the background to contribute to this science.

Sounding Rockets is a basic book covering the sounding rockets used in all countries except, and for obvious reasons, the U.S.S.R. Together with his colleagues Dr. Newell has covered rocket theory, instrumentation techniques, handling and launching techniques and the future of sounding rockets. One of the most significant tables this reviewer has seen is Table 2-2. Here are listed the results obtained from sounding rockets.

Dr. Newell states the object of study and then describes the methods used to attack the problem. Finally, the results obtained are revealed. When you consider that this was done for the determination of atmospheric pressure, density composition and temperature, winds, solar radiation, ionosphere, magnetic field of the Earth, etc., it proves to be the best synopsis and guide to the scientific attack on our lack of knowledge of our atmosphere.

Dr. Newell and his contributors have put together a commendable work. It will prove to be a most useful reference.

I. M. LEVITT
The Franklin Institute Planetarium

PROGRESS IN METAL PHYSICS, Vol. 7, edited by Bruce Chalmers and R. King. 408 pages, diagrams, 6 × 9¼ in. New York and London, Pergamon Press Ltd., 1958. Price, $16.00.

As in previous years many research workers in physical metallurgy or metal physics will find at least one article among the five contained in the 7th volume of the series which is closely related to their own work. The timing of the articles could not have been better. Al the papers describe work in areas where a great deal of activity is being displayed and where further rapid advances are expected during the next decade.

Hobstetter's article ". . . Equilibrium, Diffusion and Imperfections in Semiconductors" contains a good deal of solid state physics while the following paper by Jaffee, "The Physical Metallurgy of Titanium Alloys," is on the extreme metallurgical side. However, the articles are written in clear language so that the former will make enjoyable reading also to the metallurgist and the latter to the physicist. The remaining reviews deal with "The Thermodynamics and Kinetics of Martensitic Transformations" (Kaufman and Cohen), "The Stored Energy of Cold Work" (Titchener and Bever) and "The Properties of Metals at Low Temperatures" (Rosenberg). All three articles have in common that the subject has been reviewed very critically, and that much thought has been used to show new approaches for further research.

The new volume is, in the opinion of the reviewer, the best balanced book of the series and is fully recommended. There is an increase of 50 to 408 pages as compared with the previous volumes. This expansion seems to be responsible in part for the increase of price by 30 per cent, and many readers may find the price of $16.00 rather high. It is felt that the series may serve its purpose better by limiting the size of the single volume to 300 pages and thereby keeping the book in reach of the graduate students who are a substantial group of its readers.

H. G. F. WILSDORF
The Franklin Institute Laboratories

ON NUMERICAL APPROXIMATION, edited by Rudolph E. Langer. 462 pages, 6¼ × 9¼ in. Madison, University of Wisconsin Press, 1959. Price, $4.50.

On Numerical Approximation consists of twenty papers delivered at the Symposium on Numerical Approximation conducted by The Mathematics Research Center, United States Army, at the University of Wisconsin, April 21 to 23, 1958. The general theme was threefold: Linear Approximation, Extremal Approximation, and Algorithms. An additional paper by R. Creighton Buck, *Linear Spaces and Approximation Theory*, was included for the purpose of clarifying the role of functional analysis in its relation to present day investigations of approximation theory.

Those contributing to the symposium and consequently to this book can be identified as some of the most outstanding figures in contemporary approximation theory. It is well, then, to list their names, the titles of their papers and to add remarks where they may be appropriate.

Alexander Ostrowski, *On Trends and Problems in Numerical Approximation*

In this brief paper the stage is set for the remaining papers of the symposium. In his historical account Ostrowski discusses the tremendous influence of the digital computer on the development of approximation art and science. He also admonishes those who, in their quest for results "with a capital R," may be prone to abandon rigor.

Zdenek Kopal, *Operational Methods in Numerical Analysis Based on Rational Approximations*

The aim of this paper is to develop new methods for numerical integration and differentiation of continuous functions in terms of their tabular differences. Remarkable accuracy is obtained by rational approximation using low order differences.

Philip J. Davis, *On the Numerical Integration of Periodic Analytic Functions*

The notion that the trapezoidal rule is the "best" rule when integrating a periodic function over its period is investigated in this paper. A comparison is made between the trapizoidal and the Gaussian families to determine which is better, according to a certain norm. It is shown that the Gaussian is better if the analytic periodic function possesses isolated singularities of a certain type, sufficiently close to the interval of integration. The trapezoidal method is better for regular analytic periodic functions.

Herbert E. Salzer, *Some New Divided Difference Algorithms for Two Variables*

Preston C. Hammer, *Numerical Evaluation of Multiple Integrals*

An accounting is made of the somewhat deficient state of the art in this area of numerical approximation.

Michael Golomb and Hans F. Weinberger, *Optimal Approximation and Error Bounds*

This excellent paper displays the power and the "all-embracing" characteristic of functional analysis, whose general purpose is: "how to derive from a given amount of information about the unknown function u the total range of possible values of the functional $F(u)$."

Arthur Sard, *The Rationale of Approximation*

More on functional analysis pertaining to approximations of the sort $T(f + \delta f) \approx Ff$, where f, δf are functions, F is a given operator and T is an approximating operator. Several sets of hypotheses are formulated relative to T, F, f and δf and the results are described.

Joseph L. Walsh, *On Extremal Approximations*

Eduard L. Stiefel, *Numerical Methods of Tchebycheff Approximations*

Leslie Fox, *Minimax Methods in Table Construction*

Theodore S. Motzkin, *Existence of Essentially Nonlinear Families Suitable for Oscillatory Approximations*

I. J. Shoenberg, *In Variation Diminishing Approximation Methods*

The property "variation diminishing" is defined and several approximation methods are shown to possess it, including the Bernstein polynomials and the de la Vallee Poussin means.

Michael Golomb, *Approximation by Function of Fewer Variables*

J. C. P. Miller, *Extremal Approximations—A Summary*

This is a discussion of polynomial approximation from the practical side. Smoothing and Numerical Differentiation are discussed; also Quadrature, Tchebycheff approximation and rational approximations.

R. Creighton Buck, *Survey of Recent Russian Literature on Approximation*

This paper does not read as interestingly as the title might suggest to the casual reader of mathematical journals, if there are any. In-

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deed, it reminds this reviewer of a sophisticated edition of an article in Variety usually called "What They're Doing and Where They're Playing." Nevertheless, it reflects a great amount of research on the part of the author and an excellent bibliography of writers in the U.S.S.R. (and other countries as well) in approximation theory has resulted.

Friedrich L. Bauer, *The Quotient-Difference and Epsilon Algorithms*

J. Barkley Rosser, *Some Sufficient Conditions for the Existence of an Asymptotic Formula or an Asymptotic Expansion*

John W. Tukey, *The Estimation of (Power) Spectra and Related Quantities*

Lothar Collatz, *Approximations in Partial Differential Equations*

John Todd, *Special Polynomials in Numerical Analysis*

Because most first (and often only) courses in numerical analysis tend toward the practical and away from the theoretical, the author in this paper proposes an outline of a course of a more theoretical nature on approximation. The need for this is indicated by the existence of many books such as the one under review.

As a general observation on *On Numerical Approximation*, it must be granted that only a monumental presentation can result from such a notable lineup of contributors. It is written by experts and will be fully appreciated only by those who are equally expert or nearly equally expert. Not for browsing, it is unquestionably a warehouse of modern thinking in approximation theory.

LAURENCE C. MCGINN
The Franklin Institute Laboratories

PUBLICATIONS RECEIVED

SERVOMECHANISMS AND REGULATING SYSTEM DESIGN, by Harold Chestnut and Robert W. Mayer. Volume 1, second edition, 680 pages, diagrams, 6 × 9 in. New York, John Wiley & Sons, Inc., 1959. Price, $11.75.

SAMPLING INSPECTION TABLES, by Harold F. Dodge and Harry G. Romig. Second edition, 224 pages, 8½ × 11 in. New York, John Wiley & Sons, Inc., 1959. Price, $8.00.

A WAY OF LIFE AND OTHER SELECTED WRITINGS, by Sir William Osler. Unabridged and unaltered republication of the work first published in 1951 by Oxford University Press under the title "Selected Writings of Sir William Osler." 278 pages, illustrations, 5½ × 8 in. New York, Dover Publications, 1959. Price, $1.50 (paper).

SCIENTIFIC RUSSIAN, by George E. Condoyannis. 225 pages, 5 × 7 in. New York, John Wiley & Sons, Inc., 1959. Price, $3.50.

MANUAL OF SCIENTIFIC RUSSIAN, by Thomas F. Magner. 101 pages, 8½ × 10¾ in. New York, Prentice-Hall, 1959. Price, $3.95.

STEREO—HOW IT WORKS, by Herman Burstein. 224 pages, illustrations, 5½ × 8½ in. New York, Gernsback Library, Inc., 1959. Price: $2.90 (paper); $5.00 (hard cover).

MARS REVISITED, by Donald L. Cyr. 131 pages, illustrations, 4¾ × 7 in. Philadelphia, Dorrance & Co., Inc., 1959. Price, $3.00.

FUNDAMENTALS OF PIPE DRAFTING, by Charles H. Thompson. 66 pages, illustrations, 8½ × 11 in. New York, John Wiley & Sons, Inc., 1958. Price, $3.50 (paper).

PROCEEDINGS OF THE EIA CONFERENCE ON MAINTAINABILITY OF ELECTRONIC EQUIPMENT. 104 pages, illustrations, 8½ × 11 in. New York, Engineering Publishers, 1958. Price, $5.00 (paper).

THE PIROTECHNIA OF VANNOCCIO BIRINGUCCIO, translated from the Italian with an introduction and notes by Cyril Stanley Smith and Martha Teach Gnudi. Reissue of the 1943 printing, 477 pages, illustrations, 7½ × 10¾ in. New York, Basic Books Inc., 1959. Price, $8.50.

BOOK NOTES

FUNDAMENTALS OF ELECTRON DEVICES AND CIRCUITS, by Herman R. Weed and Wells L. Davis. 591 pages, illustrations, 6 × 9 in. New York, Prentice-Hall, Inc., 1959. Price, $9.50.

This new text was classroom tested for three years before publication. It was successfully used with both electrical and non-electrical students. Prerequisites are a working knowledge of calculus and a conventional background in electrical physics. Each of the twenty-one chapters is a unit covering one type of device or circuit. The newer devices such as transistors, magnetic amplifiers and diodes are interspersed with the older ones such as gas tubes and regulators. Each chapter ends with a set of problems which emphasize fundamental principles rather than specific applications.

THE LOGIC OF SCIENTIFIC DISCOVERY, by Karl R. Popper. Translated from the German edition by the author. 480 pages, 6 × 9¼ in. New York, Basic Books, Inc., 1959. Price, $7.50.

First published in German in 1934, this book now appears in English, translated by the author and augmented by 150 pages of new material (principally footnotes and appendices), identified by asterisks. His theme originally was that there is logic in scientific discovery and that philosophy—in spite of many opinions to the contrary—*can* pose genuine problems about the whole of scientific knowledge. Twenty-five years later, Dr. Popper, in his additions to the original work, brings up-to-date some of his earlier conceptions, without basically altering his original theme. The new material includes twelve appendices covering such topics as probability, statistics and imaginary experiments.

The final sentences in the original text sum up the author's conclusions: "Science never pursues the illusory aim of making its answers final, or even probable. Its advance is, rather, towards the infinite yet attainable aim of ever discovering new, deeper, and more general problems, and of subjecting its ever tentative answers to ever renewed and ever more rigorous tests."

CALCULATEURS ANALOGIQUES RÉPÉTITIFS, by Rajko Tomovic. 186 pages, diagrams, 6½ × 9½ in. Paris, Masson et Cie, 1958. Price, 3000 fr.

This is the first of a series of monographs to be published under the auspices of l'Association Internationale pour le Calcul analogique. The author, Director of Research at the Boris Kidrich Institute of Nuclear Sciences in Belgrade, divides his discussion of repetitive analog computers into three main parts: theory, construction and applications. In Part I (four chapters), he covers the principles of operation of a repetitive differential analyzer and the theory of error analysis. Part II (six chapters) deals with the componet parts of the computer and with installation problems, including sources of errors in the various components, both for linear and non-linear operation. Part III (three chapters) is devoted to applications for ordinary differential equations and for integral equations. A short chapter on high speed analyzers contains two interesting photographs of sets of solutions for a differential equation—one with 32 given points and the other with 32 values for one parameter.

CARCINOGENESIS BY ULTRAVIOLET LIGHT, by Harold F. Blum. 340 pages, diagrams, 5½ × 8½ in. Princeton, Princeton University Press, 1959. Price, $6.50.

This book represents over fifteen years of experimental and theoretical studies on the effect of ultraviolet light in inducing cancer. It discusses various effects of ultraviolet light on biological systems as well as experimental carcinogenesis with this agent. A description of the course of carcinogenesis is based on experimental data in which cancer is treated as a growing tissue mass. Having arrived at this description, the author does not hesitate to speculate boldly—always being careful to remain within the limits set by the experimental evidence and by modern concepts of the dynamics of living systems. A new viewpoint with regard to cancer emerges. The relation of ultraviolet light to skin cancer in man is discussed together with some practical matters concerning environmental carcinogenic agents in general.

PRODUCTION CONTROL, by Nyles V. Reinfeld. 339 pages, diagrams, 6 × 9 in. New York, Prentice-Hall, Inc., 1959. Price, $8.00.

Based on an extensive survey, this volume presents the results of seminars with more than 500 companies throughout the United States and Canada. The questions and answers emerging from round-table discussions form the nucleus of this practical and flexible guide. Viewing production control as a training ground for top management, the author shows why this premise holds true for all types and various sizes of organizations. Every phase of production control is carefully linked to management problems and methods. An up-to-date discussion of financial accountability and management is provided by the author, giving the reader invaluable background material.

OPERATIONS RESEARCH, by Maurice Sasieni, Arthur Yaspan and Lawrence Friedman. 316 pages, diagrams, 6 × 9 in. New York, John Wiley & Sons, Inc., 1959. Price, $10.25.

Much of the subject matter of this text arose from a problems course in operations research, designed as a companion course to one offered at Case Institute of Technology. This work can serve as a textbook for a one-semester introductory techniques course, at either graduate or advanced undergraduate level. The reader is assumed to have a working knowledge of the differential and integral calculus. Students majoring in such fields as industrial engineering, statistics, economics and applied mathematics will find the text invaluable.

PROGRAMMING BUSINESS COMPUTERS, by Daniel D. McCracken, Harold Weiss and Tsai-Hwa Lee. 510 pages, diagrams, 6 × 9 in. New York, John Wiley & Sons, Inc., 1959. Price, $10.25.

This volume is written primarily for those involved in the day-to-day application of electronic computers to business data processing problems. The book begins with a discussion of fundamental topics, such as the nature of the data processing problem, the central concept of the file, flow charting and the general characteristics of electronic computers. The authors employ numerous examples to explain all the standard techniques of coding. These examples are written in terms of a hypothetical computer called DATAC, which is a compilation of the features of many machines. The reader who understands the principles presented in terms of DATAC, will find no difficulty in applying them to real machines. The concluding chapters deal with a summary of the steps involved in establishing a computer application, and a critical examination of the accounting and auditing problems associated with electronic data processing.

RADIOISOTOPES FOR INDUSTRY, by Robert S. Rochlin and Warner W. Schultz. 190 pages, illustrations, 5 × 7¼ in. New York, Reinhold Publishing Corp., 1959. Price, $4.75.

The purpose of this small handbook is to show the benefits that industry can derive by the use of radioisotopes. The reader will discover techniques that can be used directly, or be stimulated in devising analogous solutions to his own problems. Chapters include introductions to the subjects of instrumentation, facilities and safety. The book is not organized in textbook form, but is intended to be useful to engineering students as well as engineers who wish to keep abreast of our advancing technology. Assuming no prior knowledge of radioactivity and nuclear science, the author aims to introduce the reader to the benefits of radioisotopes with a minimum of unfamiliar jargon or highly technical discussion. Newer applications and techniques have been described at greater length, since details on the older ones can readily be found elsewhere.

ASBESTOS: ITS INDUSTRIAL APPLICATIONS, by D. V. Rosato. 213 pages, illustrations, 5 × 7¼ in. New York, Reinhold Publishing Corp., 1959. Price, $5.75.

This text represents an accumulation of all data available on the properties, manufacturing methods and uses of asbestos in industry. Included also are the types of asbestos material available—asbestos cement, tile, asbestos heat and electrical insulation, friction materials, textiles, plastics, packings, gaskets, filters and others. A census of asbestos products is also given. The volume serves as a convenient guide to various industrial branches, research and development groups, manufacturers, and market research groups.

CURRENT TOPICS

Giant Balloon to Study Cosmic Rays.—The first in a new series of instrumented balloons was launched on June 29 by the Air Research and Development Command of the United States Air Force near Mineral Wells, Texas, in an effort to gather new data about cosmic rays near the top of the Earth's atmosphere.

The 128-ft. unmanned balloon, launched by personnel from Holloman Air Force Base under the direction of Major L. M. Bogard, Chief, Balloon Branch, Air Force Missile Development Center, carried a gondola with about 100 pounds of scientific apparatus and was designed to reach an altitude of about 100,000 ft. The instruments, developed jointly by RIAS (a research division of The Martin Company located near Baltimore, Md.) and the Air Force Cambridge Research Center (Bedford, Mass.), are designed to measure only a small portion of the cosmic particles which strike the gondola—the so-called "heavy nuclei."

Since "primary" cosmic rays disintegrate when they collide with atoms of gas in the Earth's atmosphere, producing and transferring their energy to "secondary" cosmic particles, a balloon at this altitude is expected to encounter both primary and secondary radiation. Both will be detected, but a unique device developed by RIAS will distinguish between the two by noting the change in velocity as particles pass through a beryllium plate and record them separately. Findings are radioed back to Earth.

Most of the particles in cosmic rays are the nuclei of hydrogen atoms originating from our sun and other cosmic bodies. Some of them, however, are the nuclei of heavier atoms, and these are the ones on which the current experiments will concentrate. These particular particles are believed to be especially sensitive to solar influences, and scientists will watch closely for variations in their intensity during each 10- to 20-hr. flight.

The heart of the cosmic ray experiment is a metal cylinder about the size of a portable fire extinguisher, filled with argon gas under pressure. A wire carrying a positive electrical charge runs the length of a cylinder and the plate of beryllium bisects it. When a high-speed cosmic particle enters the chamber it "ionizes" some of the gas atoms inside (that is, it knocks an electron off the outer shell of each atom). These free electrons are attracted to the wire at the center of the chamber and counted. By measuring the amount of ionization, scientists can determine the precise nature of the particle that caused it.

The balloons will be tracked continuously, and eventual recovery will be made.

New Development in Cancer Treatment.—A radiotherapy unit for the treatment of cancer in an atmosphere of high pressure oxygen has been opened at St. Thomas' Hospital, London.

The unit, which includes a pressure chamber in the form of an 8-ft. cylinder in which the patient lies, is the only one of its kind in the world. It was installed at a cost of nearly $140,000 by a British nuclear engineer-

ing firm, after development at St. Thomas' by a team consisting of radiotherapists, an anaesthetist, a pathologist, a physicist and an electronics engineer.

The team has pioneered the treatment to its present degree and is being followed by a team at University College Hospital, London. Teams in the United States, Europe and Australia are also following this lead.

Treating cancer by radiotherapy is based on the fact that all living cells can be killed by the action of ionizing radiations and there is a relationship between the sensitivity of cells to damage by them and the concentration of oxygen present.

During the time of irradiation the patient cannot be seen and nobody can remain in the treatment room. The patient is given an anaesthetic and enclosed in the pressure chamber.

Observations of the patient are made by continuous electrical readings of the heart beat, respirations, blood flow and pressure, and of brain-waves.

The patient lies on a couch which runs into the chamber on rails. Radiation is applied through the walls of the chamber. Twelve large cylinders supply oxygen. The walls of the treatment room are of dense concrete up to 2-ft. thick, the door is lead-lined and the observation window is a lead glass block 5 in. thick.

The technique has been used on more than 80 patients without mishap. The members of the British team will say no more than that they are encouraged by the results, but it is hoped the apparatus will become a permanent contribution towards the treatment of cancer.

Flat Speaker Developed.—An electro-dynamic loudspeaker of revolutionary design has been developed in the Electronics Department of the Weizmann Institute of Science in Rehovot, Israel. The main outward feature of the invention—called an Isophase speaker—is its flatness. The paper cone, which is part of any conventional speaker, has disappeared.

The new device, which is expected to be of great advantage in high fidelity, stereophonic and television systems, utilizes a membrane with a printed zig-zag winding to convert the current into vibratory motion, in place of the conventional mobile coil.

One of the main advantages of the new amplifier is that the phase of the acoustic waves is completely true and is governed by the phase of the electric current. Also, owing to its flatness, the new speaker needs much less space than the traditional type.

Multi-Tap Lumped Delay Line.—The development of a new lumped-constant Delay Line having 10 separate taps of 0.05 μsec. each providing a maximum total delay of 0.5 μsec. has been announced by Technitrol Engineering Company, Philadelphia.

Designated as CT-18, the Delay Line has an impedance of 550 ohms and a maximum rise time figure of 0.1 μsec. Attenuation is 1.0 db. max. and temperature range is from −25 C. to +85 C. Some useful applications for a delay line of this type would be as a trimmer for a long delay line (5 μsec. or more) to adjust the tolerance of the line; as a variable pulse generator with the taps controlling the width of the output pulse; to generate clock pulses; or as a variable pulse forming network.

Delay Line CT-18 is 2½ in. long, 1 in. wide and approximately 1½ in. high including threaded mounting studs on the underside of the case.

Automatic Accounting Machines.—A complete new line of automatic accounting machines that will obsolete

thousands of conventional bookkeeping machines in use today has been developed by Burroughs Corporation. Ray R. Eppert, Burroughs president, said the new series of electro-mechanical machines for business and industry represents one of the most important advances in conventional bookkeeping equipment in the past decade. Ten models of the machine were placed on the market August 1.

Designed for all types of businesses —from dress shops to public utilities— the general purpose machine is the first fully automatic mechanical accounting device equipped with a "dual printer," a special feature that increases operating speeds by as much as 50 per cent.

The new device, a numeric data processing accounting machine, Series F5000, is an outgrowth of Burroughs' famed Sensimatic. Like the Sensimatic, the new line fills the middle ground between the company's small portable bookkeeping machines and high speed electronic data processing equipment, Eppert declared.

Offering a new standard of economy and versatility, the machine is designed to improve all bookkeeping operations that require two original records, such as a statement and ledger. The machine prints both records simultaneously.

Accounts receivable, accounts payable, payroll, inventory, bank bookkeeping and posting to subsidiary ledgers are among the scores of important accounting tasks the machine will handle with greater speed and accuracy.

One of the machine's two printing heads can be de-activated for accounting functions that only require a single printing, the president explained.

All models will be equipped with a unique "program panel" that controls the automatic functions of the machine, telling it to add, subtract, retain figures for later use, print out totals and the like. The panel is capable of directing the machine to perform up to 35,000 functions without assistance from the operator.

The compact device can be "programmed" to perform four totally different accounting jobs with a simple turn of a knob.

Other features include modern styling, conventional keyboard input, controlled "gang printing" output, two to 19 memory-storage units with a maximum capacity of 228 digits, and automatic readout of data accumulated in storage units.

The machine not only will reduce the time needed to process mounting volumes of paper work, but also will permit other economies through simplification of accounting forms.

Credit unions, department stores, banks, insurance brokers, manufacturers, wholesalers, retailers, federal, state and local government agencies— virtually every business that keeps books—will be able to realize substantial savings, Eppert concluded.

Microwave Techniques to Improve Computers.—New electronic computing techniques, approaching the operating limits imposed by the speed of light, are under development by the Radio Coporation of America for use in computers that will function up to 1,000 times faster than present systems.

Announcement of the advances was made recently by E. D. Foster, Vice President and General Manager, RCA Electronic Data Processing Division, and Dr. James Hillier, Vice President, RCA Laboratories. The two RCA divisions are collaborating in an ultra-high-speed computer development study program under a contract with the U. S. Navy's Bureau of Ships.

Dr. Hillier predicted that such fu-

ture computers will be not only more compact and economical than today's systems, but will be capable of handling problems too complex for solution within a useful time by present techniques. Such problems, he added, might relate to "the frontiers of practically every science, as well as many social, economic and strategic matters."

The two executives stated that the new techniques now under development by RCA will result in "a revolutionary broadening of electronic computer functions, with profound impact upon science, industry, and defense."

"The startling advances achieved through this and similar projects will find their place during the years immediately ahead in the commercial data processing equipment which RCA will make available to commerce and industry as well as to the military services," said Mr. Foster.

Experimental use of some of the new techniques in the laboratory already has proven their ability to perform computations such as adding two digits together at speeds of 500 million times a second, according to Dr. Hillier. Stating that the techniques are based upon microwave principles employed in super-high-frequency radar and radio systems and upon new developments in solid-state electronics, he added that continuing development is expected to lead soon to "a considerable further increase" in speed.

"Since the end of World War II, we have progressed from electromechanical systems using devices capable of less than 100 basic switching operations per second to electronic computers with operating speeds more than a thousand times faster," said Dr. Hillier.

"This advance already has had major effects upon our science and technology by permitting solution of problems far too complex for the earlier systems. These problems relate to aerodynamic design, weather prediction, missile guidance, and many aspects of research in physics, chemistry, electronics and other sciences. At the same time, one of the most significant results of this increase in computing speeds has been the development and broadening application of electronic data-processing systems in commerce and industry.

"The new ultra-high-speed techniques now under development by RCA are aimed at increasing the computation rate by as much as 1,000 times beyond today's electronic computer speeds. With such an achievement, we can expect a new revolutionary effect at least equal to that resulting from the advances of the past decade."

Scope of Development Program

The research and development program on ultra-high-speed computer techniques involves scientific and engineering groups of several RCA divisions. These include not only the RCA Electronic Data Processing Division and RCA Laboratories, but also the RCA Electron Tube and RCA Semiconductor and Materials Divisions. The RCA Laboratories' contribution discussed here has consisted of a large-scale research effort involving specialists in microwave, semiconductor and computer techniques, directed by Dr. Jan A. Rajchman, an Associate Director of the Technical Staff at the David Sarnoff Research Center, Princeton, N. J.

According to Dr. Rajchman, emphasis in the research program has been placed upon new methods of performing all basic computer functions, such as switching, storing and amplifying the coded electrical signals that represent the "language" of electronic computers.

He stated that one promising method, out of several that are under study and development, involves a combination of microwave and solid-state techniques in a possible system based on the use of super-high-frequency principles. Dr. Rajchman explained that a computing system based on microwave techniques would handle information in the form of frequency pulses varying from one to ten billion cycles per second, in the following manner:

The language of the computer would consist of signals in the form of electrical oscillations at these frequencies. The oscillation in one phase represents the digit "1," while an oscillation of the same frequency in the opposite phase—that is, having a directly opposite wave pattern of peaks and valleys—would represent "0." In the binary code system used in computers, all numbers can be expressed by various combinations of these two digits, so that all numerical information could be conveyed through the system entirely by means of the two opposite phases of the high frequency signal. With the super-high-frequencies, extremely high computing speeds can thus be achieved.

"At the operating speed achieved experimentally in the laboratory, light travels only two feet in the time required for a simple computation such as the adding of two digits," said Dr. Rajchman. "Electrical impulses travel through wire at slightly less than the speed of light. Thus the speed of light, which normally might appear incredibly fast, is in this case so slow that it imposes design limitations upon our systems. The circuit connections must be made as short as possible so that the gain in operating speed is not wasted by a time lag in communicating information from one point to another in the system. It follows from this that circuits designed for computers based on microwave techniques must be miniaturized as far as possible, leading in turn to smaller as well as faster systems."